A
TREATISE
ON
SPHERICAL GEOMETRY,

CONTAINING ITS
FUNDAMENTAL PROPERTIES;
THE
DOCTRINE OF ITS LOCI;
THE
MAXIMA AND MINIMA
OF
SPHERICAL LINES AND AREAS:
WITH AN
APPLICATION OF THESE ELEMENTS TO A VARIETY
OF
PROBLEMS.

BY
JOHN HOWARD.

> *Every man has owed much to others, and ought to repay the kindneſs that he has received.*
>
> JOHNSON.

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TO THE

RIGHT REV. FATHER IN GOD,

JOHN,

LORD BISHOP OF ELPHIN, F. R. S.

MY LORD,

AS an acknowledgment of the many obligations I am under to your Lordſhip, rather than with an expectation of being able to offer any thing worthy of acceptance, I have preſumed to ſolicit your Lordſhip's name and patronage to the following ſheets, well knowing that juſt opinion the world has long ſince formed of your profound abilities in the deepeſt mathematical reſearches. Suffer me then, my Lord, to preſent you this humble ſpecimen of the produce of that tree, which, if it has produced any thing good, owes all to your generous and diſintereſted culture—and believe me to be, with the greateſt reſpect,

Your Lordſhip's much obliged,

And devoted humble ſervant,

JOHN HOWARD.

THE

PREFACE.

ADVENTURERS in unknown branches of ſcience, and travellers in unknown tracts of country, may aptly be compared; both are engaged in the purſuit of what may be ſought in vain, and which, if found, may by no means compenſate for the coſt of finding: and too oft whilſt each conſiders himſelf in the immediate graſp of the object of his wiſhes, ſome unexpected obſtacle intervenes, new Alps are to climb, and waſted labour, thought, and time, are again to be renewed.

To diſcover and arrange abſtract properties at the ſame time, is no eaſy taſk. A ſingle deſideratum (however inconſiderable) will totally put a ſtop to the progreſs of the work, whilſt an omitted datum, or awkwardly digeſted analyſis, will, as certainly, require the whole ſubſequent ſtructure to be demoliſhed. Freſh diſcoveries muſt again ſupply the deficiencies, and the pile is again to be reared upon more eligible and certain principles.

He who can do what none elſe can, diſcovers genius, ſays Lavater; and that (I fear) too oft is his only reward. In the dry abſtruſe paths of abſtract knowledge, we travel through wilds and windings, where of the few who have ability to follow, but a very ſmall part have reſolution to continue the purſuit. And hence, approbation, like reward, is confined to a very ſmall circle. Of the merit of the immortal Newton, little was known in his life-time, and of the generality who now acknowledge it, how limited are the numbers that underſtand his works.

Of the uſefulneſs of mathematical learning in general, all civilized nations are agreed; and of that part, in particular, to which theſe elements tend, the late great advances made in terreſtrial and celeſtial calculations bear ample teſtimony. From geographical conſiderations of its properties, the ſurface of the globe has been explored in almoſt every poſſible direction; and by the unremiting labours of a Herſchel, new worlds have been added to this planetary ſyſtem, and probability extended almoſt to a proof that there are an infinity of others.

Yet, whilſt ſuch important advantages have been derived from properties already aſcertained, little appears (at leaſt to me) to have been attempted towards the diſcovery of new principles, from whence other, and perhaps more important, properties might be obtained; and while the doctrine of lines *in plano*, has been generally and uniformly improving, the doctrine of the ſphere has ſcarcely made a ſingle advance from that point of improvement where the ancients left it, altho' the uniform curvature of its ſections with plane ſurfaces, the variety and harmony of its external and internal lines, and the infinite number of curves that may be delineated on its ſurface, render it a purſuit of a much more noble and extenſive nature.

Convinced of the great abilities of the leading mathematicians of the preſent age and country, I by no means lay claim to ſuperior faculties; I have only humbly attempted a freſh path in ſcience, and ſhall think myſelf amply recompenced for the labour the following ſheets have coſt me, ſhould it meet with their approbation; and what I have done on ſo intereſting a ſubject, induce ſome abler hand to improve upon the plan and extend the deſign.

The work is divided into ſix books.

BOOK I contains the fundamental principles of ſpherical angles and triangles, including not only thoſe compoſed of great circles of the ſphere; but, in general, ſuch as are compoſed of circles of leſs radii. A ſubject that I do not know has before been attempted.*

BOOK II contains the fundamental principles of ſpherical quadrangles, with ſome added properties of ſpherical triangles, and determines the meaſurement of ſolid angles.

BOOK III contains a great many curious properties of ſtraight lines, and circles drawn from given points within and without the ſurface of the ſphere, and terminating in the circumference of given ſpherical circles; alſo, ſome curious LOCI of ſpherical angles and triangles, and of lines drawn to ſpherical and cylindrical ſurfaces, analogous to ſome of the plane LOCI of Apollonius. And here the reader will find many beautiful analogies between the properties of lines drawn to meet in the ſurface of the ſphere, and of thoſe drawn to meet in the circle *in plano*.

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* In the courſe of my reſearches on this ſubject, I, in confidence, communicated many of my diſcoveries to my friends and pupils; that confidence has been abuſed, and ſeveral of theſe have already been publiſhed in various periodical publications (ſee, in particular Art. Lucubrations in ſpherics in the Mathematical Repoſitory): But whoever will compare the methods of demonſtration annexed to the ſame propoſitions, may ſoon be ſatisfied of the original ſource of diſcovery.

BOOK IV includes the doctrine of ſpherical *maxima* and *minima*; and here, I believe, will be found a variety of new and uſeful properties relative to triangles, polygons, &c. not confined to figures compoſed of great circles of the ſphere; but, in general, extending to ſuch as are compoſed of circles of leſs radii, including the remarkable problem which determines the curve, that under a given perimeter, includes the greateſt ſpherical ſurface,* as alſo an extenſive theorem of ſolid *maxima* and *minima*.

BOOK V (or I of the application) contains the conſtruction of ſpherical problems deduced from the foregoing principles; many of which will, I truſt, be found uſeful in aſtronomical reſearches: and here is included, a series of curious problems, analogous to thoſe which VIETA has conſtructed *in plano*, and that FERMAT has extended to planes and ſpheres; beginning with determining a circle on the ſphere that ſhall paſs through two given points, and touch a circle given in magnitude and poſition, and ending with finding a circle on the ſphere that ſhall touch three other circles given in magnitude and poſition.

BOOK VI (or II of the application) contains a variety of problems relating to triangles.

" It is of great uſe to the ſailor to know the length of " his line, though he cannot with it fathom all the depths " of the ocean;" † and therefore I preſume it will not be deemed impertinent to point out ſuch properties and limits as I have diſcovered in my reſearches upon the ſubject.

OB. I. The triſection and further prime diviſions of arches remaining as yet among the geometrical deſiderata, the compariſon of ſpherical lines is hence confined entirely to ſuch analogies as ariſe from continual biſection.

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* This problem is corollary I to propoſition XVI. of Lucubrations in ſpherics in the Mathematical Repoſitory, but is there wrong ſolved, as well as that propoſition.

† Locke on the human underſtanding.

II. From the infinite variety of angular ſummation in ſpherical triangles, none can be ſimilar but thoſe whoſe ſides and angles are all equal each to each; nor are theſe properties, even in triangles compoſed of great circles, always ſufficient to prove ſimilarity. For the two equal triangles into which an iſoſceles ſpherical triangle may be divided, though alike in every other reſpect, have not the required property of coincidence; but when the data are applied to thoſe compoſed of circles of different magnitude, it will be neceſſary alſo to prove that the centers of the equal circles of which they are compoſed, have their radii lying on the ſame ſide of the adjacent ſides or angles in each. (See T. 3. 4. 5. 15. and 18. B. I.)

III. From theſe premiſes it would ſeem that the conſtructions of ſuch problems, or demonſtrations of ſuch theorems in ſpherics as are ſimilar to thoſe which, *in plano*, are ſolved by the application of proportionality or ſimilarity, would be impracticable. The contrary, however, is particularly inſtanced in the ſolutions of P. 20. and 21. B. I. (of the conſtruction of ſpherical problems) being ſimilar to P. 15. and 48. of the application in Profeſſor Simpſon's Geometry; and alſo of T. 24. and 25. B. I. which are ſimilar to T. 31. and 34. B. II. of Mr. Emerſon's Geometry, the ſolution of which theorems alſo hints another very uſeful obſervation.

IV. That the ſolutions of ſpherical problems may often be greatly ſimplified by determining thoſe lines or points wherein the planes of circles meet each other, by which artifice they may then be reduced to thoſe of plane problems.

(See alſo L. 1. and 2. B. I. and P. 16. and 17. B. I. of the conſtruction of ſpherical problems.)

V. As no two great circles can be parallels, it would alſo ſeem that problems of the ſphere, ſimilar to ſuch as are ſolved *in plano*, by ſuch means would alſo be inſoluble; but this deficiency may alſo be ſupplied, an inſtance of which occurs in T. 24. B. I. which is ſimilar to T. 31. B. II. of Mr. Emerſon's Geometry, mentioned before.

VI. It has been explained (OB. II.) that the ſimilarity of ſpherical triangles can only be proved when both equality and coincidence are included in the data; the converſe, however, viz. the equality of ſpherical triangles, whoſe equal ſides and angles are oppoſitely ſituated, may clearly be demonſtrated; for it is proved in T. 10. B. II. that triangles compoſed of arches of different circles, may, under ſuch circumſtances, be equal; and in great circle triangles not only generally ſo, but that the applied angles will alſo be equal. (See T. 6. B. II. and its corollaries.)

VII. The indefinitely varied ſummation of ſpherical angles (mentioned in OB. II) has however its peculiar advantages; for hence the angles alone determine the triangle; (T. 18. and 19. B. I.) and as geometricians have aſſumed arches for the meaſures of angles, theſe in great circle triangles compoſe the ſides of other ſupplementary triangles, (T. 1. B. II. of the conſtruction of ſpherical problems) or the contrary; by which means the data may often be reduced to others, more ſimple and elegant. (See P. 2. 3. and 8. B. II. of conſtruction.)

VIII. Again, as any two ſides of a great circle ſpherical triangle will, when produced, meet again at a ſemicircle's diſtance from their interſection; the ſuppoſing theſe arches produced to this point, will often lead to an elegant analyſis, (See P. 1. and 3. B. II. of conſtruction, and alſo P. 6. of the ſame, which would otherwiſe require a LOCUS not geometrically determinate on the ſphere.)

IX. The ſimilarity of analyſis between many problems *in plano*, and like ones on the ſphere, ſo often recurs, that every geometrician will naturally be led to attempt ſolutions by tracing that analogy; I therefore preſume it is ſufficient that it is mentioned.

It may be urged as an objection to ſome of the propoſitions, that they are demonſtrated in too general a manner; and I confeſs, that by a diviſion of many of them into their particular caſes, (after the manner of Euclid and Apollonius) the corollaries reſulting from thence would be conſiderably enlarged; but then the work would have been in proportion much more bulky and expenſive: the ſame may alſo be ſaid of the diagrams, already very numerous. He, however, who may be inclined to purſue the ſubject in that manner, will (I have no doubt) find his labour amply repaid, by the diſcovery of many beautiful PORISMS, at preſent concealed in each common maſs.

To urge, in excuſe for imperfection, the plea of misfortune, may be deemed equally as impertinent as it would likely be unavailing. Conſcious as I am that the recital of individual evils can ſeldom promote general good, it is of little moment to tell mankind that this work has been brought to its preſent ſtate "amidſt inconvenience and diſ-"traction, in ſickneſs and in ſorrow;" that it had its origin in a vain endeavour to amuſe the mind under a violent and lengthened preſſure of bodily pain; or that it has been interrupted in its progreſs by a concatenation of private ills, ſuch as (I hope) is not commonly the lot of humanity. I only and humbly requeſt, that as theſe circumſtances explain the reaſon of its delay, they may alſo apologize for me to many of my candid and worthy Subſcribers, for having ſo long procraſtinated their wiſhes. The work, ſuch as it is, I deliver to the world, to ſtand or fall by its intrinſic merit, convinced, whatever may be its fate, that I have endeavoured well.

Newcastle upon Tyne,
March 10th, 1798.

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THE

ELEMENTS

OF

SPHERICAL GEOMETRY.

BOOK I.

DEFINITIONS.

1. GEOMETRY is that ſcience by which we compare ſuch quantities together as have extenſion.

Extenſion is diſtinguiſhed into length, breadth, and thickneſs.

2. A point is that which hath no parts, or which hath no magnitude.

3. A line is length without breadth.

The terms, or bounds, or extremes of

a line, are points. A ſtraight line lies evenly between its extreme points.

4. A ſurface is that which hath length and breadth only.

5. A plane ſurface is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies.

6. A plane angle is the inclination of two ſtraight lines which meet in a point.

7. A plane right angle is that which is made by one right line falling upon another, and making the angles on each ſide equal.

8. An obtuſe plane angle is greater than a right angle.

9. An acute plane angle is leſs than a right angle.

10. A circle is a plane figure bounded by one line equidiſtant from a fixed point called its centre.

11. The deſcribed line is called its circumference.

12. A ſtraight line drawn from the centre to any point in the circumference is called its radius.

13. A ſtraight line drawn through the centre, and terminating in the circumference each way, is called a diameter.

14. Either of the ſpaces bounded by this line, and the portion of the circumference cut off thereby, is called a ſemicircle.

15. The doctrine of the ſphere is the geometry of ſpherical figures.

16. A ſphere A I B L i is formed by the revolution of a ſemicircle A I B

round its diameter, A B at reſt.—(Pl. I. fig. I.)

17. The centre O of the ſphere A I B L i is that point which is equidiſtant from the extremes A, B, of the diameter A B.

18. A ſpherical circle I L i is formed by a point I in the circumference A I B, of the generating ſemicircle A I B, revolving round its fixed diameter A B.

19. If the point I be at an equal diſtance from each extremity of the circumference A I B, then will it generate, what is called a great circle, L i I of the ſphere.

20. At all other points I, it will generate what is called a leſs circle *L i I* of the ſphere.

21. The pole A or B of a ſpherical circle *L i I*, is a point on the ſurface of

the ſphere, equidiſtant from every point, *L i I* of its circumference.

22. The neareſt pole A, I call its centre.

23. An arch of a circle I i, is any portion formed by the generating point I.

24. The chord or ſubtenſe of an arch I i, is a right line I i joining the two extremes of that arch.

25. The radius A I, is an arch of a great circle drawn from the centre A, of a circle I i L, to its circumference; and the diameter L A I, is the radius A I produced till it paſs through the oppoſite ſide L of the circle, from its centre A, and is therefore equal to twice the radius.

26. A tangent *I* V, is an arch of any circle touching another circle *L i I*, in

a point *l*, and which, if produced, will not cut it.

27. A ſpherical angle L V D, is the inclination of two circles L V, V D of the ſphere to each other, interſecting in a point V, called the angular point.

28. If the planes A O B i, L V I O, of two great circles, are perpendicular to each other; the angle B i I formed by their interſection, is called a right angle; and the arches A i B, L i I, are ſaid to be perpendicular to each other.

29. If the planes V D O, L O V, of two great circles, V D, L V, are not perpendicular to each other; the angle L V D, formed by their leſs interſection, is called an acute angle; and the angle D V I, formed by their greateſt inclination, is called an obtuſe angle.

30. Equal and ſimilar ſpherical angles

are ſuch as, if laid upon each other, coincide in every part.

31. Two circles on the ſphere are ſaid to cut each other, when they fall partly within and partly without one another; or, when their circumferences cut each other.

32. A ſpherical triangle is made by the interſection of three circles on the ſphere. If compoſed of great circles, it is called a great circle triangle.

33. A right angled ſpherical triangle is compoſed of three great circles; two of which form a right angle.

34. An iſoſceles ſpherical triangle has two of its ſides equal in length and curvature, and mutually concave or convex to each other.

35. An equilateral ſpherical triangle hath all its ſides equal in length and

curvature, and mutually concave or convex to each other.

36. An oblique ſpherical triangle hath all its ſides unequal.

37. Equal and ſimilar ſpherical triangles are ſuch, that if placed upon each other, will coincide in every part.

38. By circles or arches of the ſame, or equal curvature, I mean ſuch as are of equal radii.

39. Spherical arches are ſaid to be equal, when, if applied to each other, they coincide in all their parts.

40. Spherical triangles are ſaid to be of uniform curvature when formed by arches, whoſe radii are all equal.

41. A great circle of the ſphere B i A is perpendicular to any other circle L i I,

when its plane A O i B is perpendicular to the plane L Q I i of that circle. — (Pl. 1. fig. 2.)

42. All ſtraight lines, L O, O I, &c. drawn from the centre O to the ſurface of the ſphere are equal. This follows from definition 16.

AXIOMS.

1. THINGS equal to one and the ſame thing, are alſo equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If equal things be added to equal things, the wholes will be equal.

5. If from equal things, equal things be taken away, the remainders will be equal.

6. If equal things are multiplied by equal things, the products will be equal.

7. If equal things are divided by equal things, the quotients will be equal.

8. If to or from unequal things, equal things be added or taken away, the ſums or remainders will have the ſame difference as the unequal things firſt propoſed.

9. Two ſtraight lines cannot include a ſpace.

10. Two equal arches of equal circles having the ſame centre, cannot include a ſpace.

POSTULATES.

1. THAT from any given point to any other, a ſtraight line may be drawn.

2. That from any given point on the ſphere to any other given point on the ſphere, any arch of a circle may be drawn.

3. That any arch of a circle may be produced till it becomes a circle.

4. That from any point as a centre, with a radius equal to a given right line on a plane, or a given arch on the ſurface of the ſphere, a circle may be deſcribed.

NOTES AND OBSERVATIONS.

A PROPOSITION is, when ſomething is propoſed to be done.

A **THEOREM** is, when ſomething is propoſed to be demonſtrated.

A **LEMMA** is, when ſome premiſe is demonſtrated, in order to render the thing in hand more eaſy.

A **COROLLARY** is, a conſequent truth, gained from ſome preceding truth.

A **SCHOLIUM** is, when remarks and obſervations are made upon ſomething going before.

EXPLANATION OF THE SIGNS.

The ſign =, denotes that the things betwixt which it ſtands are equal.

The ſign +, denotes that the quantity which it precedes is to be added.

The ſign −, denotes that the quantity

which it precedes is to be taken away or ſubtracted.

The ſign ±, denotes that the quantity which it precedes may either be added or ſubtracted.

Ob. I. Wherever the word arch occurs in the following work, it is to be underſtood a portion of a great circle on the ſphere, unleſs there differently ſpecified.

Ob. II. The initial letters (D. T. P. C. E. &c.) included within a parentheſis, are intended to repreſent the terms, Definition, Propoſition, Theorem, &c. thus (D. 5. B. I.) ſignify Definition the fifth, Book the firſt; and (C. II. T. I. B. II.) mean Corollary ſecond, Theorem firſt, Book ſecond; alſo, (4. I. E.) the fourth propoſition of the firſt book of Euclid, &c.

THEOREM I.

THE ſection of a ſphere L A I B, and a plane I Q L i, is a circle L i I. (Pl. 1. fig. 2.)

Through O, the centre of the ſphere, draw the ſtraight line O Q perpendicular to the plane L i I, and meeting the ſphere in A and B. Alſo through Q in the plane L i I Q, draw any ſtraight line, L Q I, meeting the ſphere in L I, and to any other point i, where the plane L i I meets the ſphere, draw Q i; alſo draw L O, i O, and I O.

Becauſe the points L i I lie all in the ſame plane to which O Q is perpendicular, in Q the triangles L Q O, i Q O, and I Q O, have the angles at Q, all right ones (converſe 5. xi E) and ſince L O=i O=I O (D. 42, B I) and Q O common, they are equal and ſim. (4. I. E) and, conſequently, L Q=i Q=I Q. Therefore L i I is a circle, whoſe centre is Q. (D. 10. B. I.)

COR. I. When the point Q coincides with O, the ſection L i I is a great circle.—(Pl. 1. fig. 3.)

Suppoſe A i B the generating ſemicircle paſſing through i, and draw the ſtraight lines A i and B i. The triangles A O i, B O i, having their ſides A O=B O and O i common, and the angles at O both right, have the ſides A i =B i (4. I. E.) and hence alſo the arch A i= B i. (28. III. E.) And ſuppoſing the generating circle removed to any other poſition, A I B, it may in the ſame manner be proved that the arch A I=B I. Whence the generating points I, i, &c. being in all caſes at equal diſtances from the poles A B, the circumference L i I will be a great circle. (D. 19. B. I.)

COR. II. The plane of every great circle L i I paſſes through the centre O of the ſphere.

COR. III. If a great circle L A I B be perpendicular to any other circle L i I, it paſſes through its poles A B. —(Pl. 1. fig. 2.)

COR. IV. The line Q O is perpendicular to the plane L i I.

COR. V. A ſpherical ſurface A i I is every where equi-diſtant from the centre O of the ſphere.

COR. VI. If Q be the centre of any circle of the ſphere, and A Q be perpendicular to its plane, L i I meeting the ſphere in A, then will A be its pole.

Draw the arch A i. Since the angle L O A =i O A=I O A, the arches A L, A i, and A I, are all equal. (26. III. E.) And hence A is the pole of L i I. (D. 21. B. I.)

COR. VII. The ſtraight line L I formed by the interſection of the planes of any two great circles, L A I, L b I, paſſes through O, the centre of the ſphere.—(Pl. 1. fig. 3.)

COR. VIII. Great circles, A i B, A I B, mutually biſect each other.

For their line A B of common interſection, paſſes through the centre O (C. II. T. I. B. I.) and conſequently makes each of the arches, A i B, A I B, a ſemicircle. (D. 16. B. I.)

PROBLEM I.

Through any two points, I, b, on the ſphere, to deſcribe a great circle.—(Pl. 1. fig. 3.)

From I and b to the centre O of the ſphere, draw I O and b O reſpectively, and produce one of them, b O, to meet the ſuperficies on the oppoſite ſide at a. Laſtly, through the points I b a (5. IV. E.) deſcribe a circle, and the thing is done.

By conſtruction, the points I b a are all in the ſurface of the ſphere, and alſo in the ſame plane, paſſing through its centre O. Wherefore, b a is a great circle. (C. II. T. I. B. I.)

COR. I. Only one great circle, I b a, can paſs through two points, I, b, that are not a ſemicircle diſtant upon the ſphere:

For no plane but one, I O b, can paſs through three given points except they lie in a ſtraight line.

PROBLEM II.

Upon a given arch A B on the ſphere, to deſcribe an equilateral triangle A B C.—(Pl. 1. fig. 4.)

From the centre A, with the radius A B, deſcribe the circle B C. Alſo from the centre B, with radius B A, deſcribe the circle A C. Laſtly, (by P. I. B. I.) through C, where theſe circles meet, and A, B, deſcribe the great circles A C and B C reſpectively, and the thing is done.

Becauſe A is the pole of the circle B C, the arch A B=AC; and by the ſame reaſoning, A B=B C. Hence (A. 1.) A B=A C=B C. Whence (D. 35. B. I. and C. I. P. I. B. I.) A B C is an equilateral triangle.

PROBLEM III.

From a given point A on the ſphere, to draw an arch A L equal to a given arch B C.—(Pl. 1. fig. 9.)

From the point A to B draw the arch A B, and upon it deſcribe the equilateral triangle D A B, (P. II. B. I.) and produce the great circles D A, D B to L and G; from the centre B, at the diſtance B C, deſcribe the circle C G; and from the centre D, with radius D G, deſcribe the circle G L. Then is A L=B C.

Becauſe the point B is the pole of C G, B C is=B G; (D. 21) and becauſe D is the pole of G L, D L=D G, (D. 21) and D A=D B by conſtruction. Therefore (A. 1. and 5.) A L=B G=B C.

PROBLEM IV.

From the greater of two given arches AB and C, to cut off a part equal to the leſs C.—(Pl. 1. fig. 10.)

From the point A draw the arch AD equal to C; (P. III. B. I) and from the centre A, and at the diſtance AD, deſcribe the circle DE. Then, (D. 21. B. I.) AE=AD=C by conſtruction.

PROBLEM V.

To biſect a given arch AIB of any circle AIB on the ſphere.—(Pl. 1. fig. 11.)

Compleat the circle AIB, and draw the chord AB. From the centre O of the circle AIB, on the plane AIB, let fall OQ perpendicular to AB. Then, OQ produced will meet the circumference in I, the point required.

Join A, O, and B, O. The triangles AOQ, OQB, are equal and ſimilar, (4. I. E.) and angle AOQ=BOQ; and hence the arch AI =BI. (26. III. E.)

COR. I. Draw O V perpendicular to the plane I B O A, meeting the ſphere in V. Then is V the pole of the circle A I B. (C. VI. T. I. B. I.)

COR. II. Hence all great circles, V I, V A, &c. whoſe planes, V I O, V A O, are perpendicular to the plane A I B O of any circle paſs through its pole V.

THEOREM II.

Only one circle A L B of the ſame curvature can paſs through two given points A, B, on the ſphere, having its centre O on the ſame ſide of thoſe points.—(Pl. 1. fig. 12.)

For ſuppoſe any other circle A I B paſſing through A and B, having its centre on the ſame ſide of A and B: Alſo to any point L, draw the arch O L, meeting the circle A I B in I, and join A, O, and O, B: Then, becauſe the points A, B, are in both the circumferences A L B, A I B; and AO=O B=O L, are radii of both circles by ſuppoſition; O is their common pole. (D. 21.) That is OL=OI±IL=OI; the leſs to the greater, which is abſurd.

COR. I. No circle on the ſphere can cut another in more than two points:

Otherwiſe, three equal arches might be drawn to the circumference from a point without the centre, which is abſurd.

PROBLEM VI.

Through two given points A, B, on the ſphere ABC, to draw an arch AB of given radius.—(Pl. 1. fig. 13.)

With radius AO, equal to the given one, and centre A, deſcribe the circle OV. Alſo with equal radius BO, and centre B, deſcribe the circle OI, cutting the former in O. About O as a pole, with radius OB, deſcribe the circle BIVO, and it is done.

Join A, O, (P. I. B. I.) then AO=OB by conſtruction, both radii of the required circle.

OB. 1. Becauſe the circles OV and OI may interſect each other in two points O, o, the problem admits of two ſolutions; and when they neither cut or touch each other, the ſolution becomes impoſſible.

Ob. II. Hence alſo in all conſtructions to which this problem is applied, it muſt be ſuppoſed to be known on which ſide of a great circle paſſing through the points A, B, the centre O or o, is to lie.

THEOREM III.

If two ſpherical triangles, A B C, a b c, have two adjacent ſides, A B equal to a b, A C equal to a c, and the included angles A, a, equal in each; alſo the third ſides B C, b c, of equal radii, whoſe centres lie on the ſame ſide of the points B, C, and b, c, reſpectively; theſe triangles and their correſponding parts ſhall be equal.—(Pl. 1. fig. 14. 15.)

For ſince angle A=a, and A B=a b, &c. therefore if A be laid upon a, ſo that A B fall upon a b; then A C will fall upon a c; the point B will fall upon b; and C coincide with c. Whence the baſe B C muſt coincide with b c. (T. II. B. I.) and, conſequently, the whole triangles are equal. (D. 37.)

COR. I. If the ſides C A, A B, and c a, a b, be of equal curvature, forming each one continued arch C A B = c a b; then will the lunar areas C A B, c a b, alſo be equal.—(Pl. 1. fig. 5. 6.)

THEOREM IV.

If two ſpherical triangles, A B C, a b c, have one ſide B C equal to one ſide b c, and the adjacent angles equal; B equal to b, and C to c; and the remaining ſides A B, a b, and A C, a c, of equal curvature in each pair. That is, A B to a b, and A C to a c; alſo having the centres of each correſponding ſide A B, a b, and A C, a c, lying on the ſame ſide of their reſpective extremes; theſe triangles will in all their parts be equal.—(Pl. 1. fig. 14. 15.)

For if A C be leſs than a c, lay off g c on a c, equal to A C, (P. 4. B. I.) and join g b. Then, becauſe b c = B C, and angle c = C. The triangles A B C, g b c, are equal; (T. III. B. I.)

and angle g b c=B=c b a, the leſs to the greater, which is abſurd.

COR. I. If the ſides B A, A C, and b a, a c, be of equal curvature, forming each one continued arch B A C and b a c, then will the lunar areas B A C, b a c, alſo be equal.—(Pl. 1. fig. 5. 6.)

THEOREM V.

If two ſpherical triangles, A B C, a b c, have all their ſides reſpectively of equal length and curvature, A B to a b, A C to b c, and B C to b c; alſo having the centres of each correſponding ſide, A B, a b, A C, a c, and B C, b c, lying on the ſame ſide of their reſpective extremes; all their reſpective angles will be equal, and the wholes equal.—(Pl. 1. fig. 14. 15.)

Draw the chords A B, A C, B C, a b, a c, b c.

Becauſe the arch A B=a b, the chord A B =a b; and in like manner the chord A C =a c. and B C=b c; whence if the baſe B C be laid upon b c, the remaining parts of the plane triangles A B C, a b c, will coincide;

and the point A on a, (8. I. E.) and, conſequently, the arch B A coincides with b a; (D. 37.) and A C with a c.

COR. I. From two given points B, C, on the ſphere, no two arches equal to A B, A C, and of the ſame curvature can be drawn on the ſame ſide of B C, to meet in any other point beſides A.

COR. II. If the ſides B A, A C, and b a, a c, be of equal curvature, forming each one continued arch B A C = b a c, then will the lunar areas A B C, a b c, be alſo equal.—(Pl. 1. fig. 5. 6.)

THEOREM VI.

If two equal circles L A I, C A I, cut each other on the ſphere in A, I, then are the angles at A and I equal.—(Pl. 1. fig. 16.)

Biſect the equal arches A I in D and E, (P. V. B. I.) and draw the great circle D E. (P. I. B. I.) Then, ſince the ſide A D = I E, A E = D I, and D E common to the triangles D A E, D I E, they are equal, (T. V. B. I.) and angle A = I.

COR. I. Lay off A E V=A D C, (P. IV. B. I.) and through A V and A C draw circles A V F, A C F, of equal curvature with L A I and C A I, then is the angle V A I=C A I.

For angle V A E=C A D. (T. VI. B. I.) From each of theſe equals take angle D A E, and there will remain V A I=C A I.

COR. II. If two equal circles L A I, C A I, cut each other on the ſphere, the angles V A I, V I A, C A I, A I C, of the equal lunes V v A, C c A, made by their interſection, will all be equal.

Suppoſe the equal arches A V v, A C c, ſo taken, that the circles A v, A c, drawn through theſe points reſpectively, ſhall each coincide with its circle A v and A c; then the laſt Cor. being generally true, muſt alſo be ſo for any particular caſe.

THEOREM VII.

Let I A V, I a V, be any two equal circles on the ſphere meeting in I, V, and let there be laid off upon the equal arches I A V, I a V; the arch I B equal to I b, and I A equal to I a; and through I V draw the ſtraight line V I C: Then, ſtraight lines drawn through A, B, and a, b, reſpectively, will meet V I C in the ſame point C.—(Pl. 1. fig. 18.)

If the arch I a V be applied to I A V; then, becauſe, I B = I b and I A = I a, the points B, b, and A, a, will reſpectively coincide; (T. II. B. I.) and, conſequently, the ſtraight lines A C, a c: Alſo, ſince the lines V I C, A B C, are in the plane of the ſame circle; they muſt meet ſomewhere in V I C, and ſince the lines A B C, a b c, coincide, they muſt meet V I C in the ſame point C.

COR. I. Join the oppoſite points A, b, a, B, and draw the ſtraight lines B b, A a: Then, are the angles B, b, and A, a, reſpectively equal, and B b parallel to A a; and alſo, A b = a B.

This is evident from the Theorem, and 4. 1. E.

COR. II. Join V, B; V, b; A, V; a, V, with ſtraight lines: Then, will the plane triangles V B a, V b A, be equal and ſimilar.

The arches V A B, V a b; and alſo, V A, V a, being reſpectively equal; and A b = a b, (C. I.) the whole triangles are equal. (21. 3. and 26. 1. E.)

COR. III. Hence, if the ſides A V, V B; V b, V a; A b, a B, be reſpectively equal in the two plane triangles, A V b, a V B; and the arches belonging to them, of uniform curvature, having the centres of their radii, in each pair, lying on the ſame ſide of their reſpective extreme points; the ſpherical angles A V b and a V B will alſo be equal. —(Pl. 1. fig. 7. 8.)

COR. IV. Hence, to lay off a ſpherical angle a V B equal to a given angle A V b, compoſed of equal circles. From the point V, lay off any two chords, A V, V b, on the ſides A V, V b, containing the given angle, and join A, b; make the triangle a V B = A V b; then, if circles of the ſame curvature be ſimilarly drawn through a V, V B, (P. VI. B. I.) it is done.

OB. This Cor. will alſo hold good when the arches V v A, V v B, are convex to the arches V b, V a, ſince the angle A V v=B V v, (T. VI. B. I.) taken from the equal angles V, V, will leave the angle v V b=v V a.

COR. V. If the arches A V, V B, &c. be reſpectively equal, each to each, in the ſpherical triangles A V b, a V B, and of uniform curvature, having alſo their reſpective centers as in

COR. III. then will the reſpective angles be equal.

THEOREM VIII.

The angles C and B, at the baſe of an iſoſceles triangle B A C of uniform curvature, are equal.—(Pl. 1. fig. 26.)

Draw the chords A B, A C, B C. Since the plane angle B = C, and the ſides that compoſe them, (A B = A C and B C = B C) are equal, (5. I. E.) the ſpherical angles B and C will be equal. (C. III. T. VII. B. I.)

COR. I. If the angles B and C, at the baſe, be equal, and the triangle of uniform curvature, the ſides A B, A C, will be equal.

COR. II. An equilateral ſpherical triangle is alſo equiangular and the contrary.

THEOREM IX.

Any two ſides A B + B C of a ſpherical triangle A B C, compoſed of great circles, are greater than the third ſide A C. —(Pl. 1. fig. 19.)

Let D be the centre of the ſphere; join D A, D B, D C. The ſolid angle at D is compoſed of the three plane angles A D B, A D C, B D C, and (by 20. 11. E.) any two of them, A D B, B D C, are greater than the third A D C; that is any two ſides A B, B C, are greater than the third ſide A C.

COR. I. An arch of a great circle is the ſhorteſt diſtance between any two points on the ſurface of the ſphere.

THEOREM X.

In any ſpherical triangle compoſed of great circles, the greateſt ſide is oppoſite to the greateſt angle and converſely.— (Pl. 1. fig. 17.)

Suppoſe A, the greateſt angle in the triangle B A C, and lay off the angle B A D=B. (C. IV. T. VII. B. I.) Then will B D=A D. (C. I. T. VIII. B. I.) Therefore, A D+D C= B C. Now, A D+D C is greater than A C, (T. IX. B. I.) whence B C is greater than A C.

Again, in the ſpherical triangle B A C, the greateſt angle A is oppoſite the greateſt ſide B C.

If it be denied, let the angle C be ſuppoſed greater than A.

Becauſe the angle C is greater than A; the ſide B A is greater than B C; but it is alſo leſs by ſuppoſition, which is abſurd.

THEOREM XI.

If one angle L i A formed by the interſection of two great circles, L i I, A i B, be a right angle; all the remaining angles A i I, I i B, L i B, formed by them about the point i, are alſo right angles.—(Pl. I. fig. I.)

Draw i O to the centre O of the ſphere, which will lie in the common interſection of the planes L i I O, A i B O, of theſe circles. (C. II. T. I. B. I.) By D. 28. the plane L O i is perpendicular to the plane A O i. But the plane A O i is alſo perpendicular to the plane i O I; (ſince i O I is in the ſame plane with L O i) therefore A i I is alſo a right angle. And in like manner it may be proved that I i B and L i B are right angles.

COR. I. If the angles L i A, A i I, which one great circle A i B makes with another L i I, upon the ſame ſide of L i I, be equal, then are each of them right angles.

PROBLEM VII.

From a given point C in a great circle A C B, to erect a perpendicular great circle C P.—(Pl. 1. fig. 22.)

From C lay off A C=C B (P. IV. B. I.) and upon A B, deſcribe an equilateral triangle A P B. (P. II. B. I.) Through P, C, draw the arch P C; (P. I. B. I.) and the thing is done.

Becauſe A C=C B, and A P=P B, and alſo P C common to the two triangles A P C, P B C, the angles A C P, B C P, are equal, (C. V. T. VII.) and each a right angle; (C. I. T. XI. B. I.) conſequently P C is perpendicular to A B. (D. 28.)

COR. I. If P C be perpendicular to A B, then are A C P, P C B, each right angles.

THEOREM XII.

If to any point C in a great circle A B, ſeveral other great circles D C, E C, are drawn on the ſame ſide, all the angles A C D+D C E+E C B, formed at the point C, taken together, are equal to two right angles.—(Pl. 1. fig. 20.)

Draw P C perpendicular to A B. (P. VII. B. I.) Then, ſince A C P+P C B are equal to two right angles, (C. I. P. VII. B. I.) and the angles A C D+D C E+E C B occupy the ſame ſpace, they are alſo equal to two right angles.

COR. I. All the angles about one point on a ſphere, taken together, are equal to four right angles.

COR. II. If all the angles at C on one ſide of A B be equal to two right angles, then the points A, C, B, are all in one great circle.

THEOREM XIII.

If two great circles A E B, D G C, cut each other, the oppoſite angles E, G, will be equal.—(Pl. 1. fig. 25.)

For A E C+E=two right angles, (T. XII. B. I.) and A E C+G=two right angles: (ibid.) Hence A E C+E=A E C+G, (A. 1.) and E=G: (A. 5.) And hence alſo A G C=B G D.

COR. I. If A B be a great circle, and E=G, then is C D a great circle.

THEOREM XIV.

If C I D, C E D, be equal circles on the ſphere, and from C, D, their mutual interſection, there be laid off C I equal to D E; then a great circle paſſing through the points E, I, will make the alternate angles C I E equal to D E I.—(Pl. 1. fig. 21.)

Since C I=D E, and C I D=C E D, by taking the former equals from the latter, we have C E=I D; (A. 5.) and ſince E I is common to the two triangles C I E, D E I, they are equal and ſimilar. (T. V. B. I.)

COR. I. If C I D, C E D, be great circles, the arches D I+D E are equal to the ſemicircle D C.—(Pl. 1. fig. 30.)

COR. II. Produce E I to V, and draw D V. (P. I. B. I.) Then, if C I D, C E D, be great circles, the triangles D E V, D I V, have the angles V E D, (=C I E) =D I V, and the angle V, and ſide D V common, and yet are unequal.

PROBLEM VIII.

From a given point P on the ſphere, to draw a great circle O P C perpendicular to another given circle D C.—(Pl. 1. fi. 2.)

Through O, the pole of the given circle and the given point P, draw the great circle O P C, (P. I. B. I.) meeting the circumference in C, and the thing is done.

The demonſtration is evident, from C. II. P. V. and D. 41.

THEOREM XV.

If through the neareſt poles P, Z, of two equal circles D V I, L K C, a great circle P V K Z be drawn, and the diſtance P Z biſected in O, then any great circle L O I paſſing through O, and cutting the circles in L and I, will make the alternate angles equal, viz. V I L equal to K L I.—(Pl. 1. fig. 28.)

Becauſe the angles L O K = I O V, (T. XIII. B. I.) and V O = O K; and alſo theſe arches both perpendicular at V and K reſpectively, (P. VIII. B. I.) the triangles V O I, L O K, are equal and ſimilar (T. IV. B. I.); and conſequently the angle V I O = K L O.

COR. I. If the circles D V I, L K C, are alſo parallel, then any great circle L O I drawn to cut them in I and L reſpectively, will make the alternate angles D I L, C L I, equal.—(Pl. 1. fig. 27.)

COR. II. Lay off L R = I A on the ſame ſide of L I; then a great circle R A drawn through theſe points, will make the angle R A I = V I L, and A R = I L.

From I and A upon L C, let fall the perpendiculars I B, A C, reſpectively; then, ſince I B = A C by conſtruction, and L R = I A = B C, by adding R B to each of theſe equals, we have L B = R C; alſo the arches I L, R A, being great circles, and the angle L B I = A C R, the angle L I B is equal R A C, (T. III. B. I.) and I L = A R. Whence (A. 5.) L I V = R A I.

COR. III. If L I, R A, be any arches of equal curvature, having their centres on the ſame ſide, then will the angles L I V, R A I, always remain equal.

COR. IV. Any great circle L A, cutting the equal parallels L C, D A, will make the alternate angles C L A, V A L, equal.

For laying off I A = L R, and deſcribing the arches L I, R A, then is C L A = V A L. (T. V. B. I.)

THEOREM XVI.

If two ſpherical triangles A B C, a b c, have one ſide A B equal to one ſide a b, (D. 39); and the oppoſite angles equal B to b, and C to c, and the ſide a c, not the ſupplement of A C, to a ſemicircle; theſe triangles and their correſponding parts will be equal.—(Pl. I. fig. 14, 15.)

If b c be ſuppoſed leſs than B C, lay off on B C, B H = b c, (P. IV. B. I.) and join A H. (P. VI. B. I.)

Then becauſe b a=B A, b c=B H, and angle B=b. The triangles a b c, A B H, are equal and ſimilar, (T. III. B. I.) and angle c=B H A; but B H A is not equal angle C by the converſe of C. II. T. XIV. B. I. and by ſuppoſition; therefore angle c is not equal angle C; which is contrary to the ſuppoſition.

THEOREM XVII.

All circles A G F, A V D, on the ſphere, whoſe centres I, C, lie in the ſame arch A C, and whoſe circumferences paſs through the ſame point A in that arch; touch each other at that point only.— (Pl. 1. fig. 29.)

If it be denied, let them alſo touch in O. And, ſuppoſe their centres I, C, to lie both on the ſame ſide of the point A. Join C, I; O, I. (P. I. B. I.)

Becauſe C is the centre of the circle A V; C A=C O, and for the ſame reaſon A I=I O. Therefore, C O+C I=A C+C I=A I=I O, which is abſurd. (T. IX. B. I.)

Again, ſuppoſe their centres I, C, to lie on different ſides of the point A; (Pl. 2. fig. 1.)

then, ſince as before, C A = C O and A I = I O, we obtain C I = C A + A I = C O + I O, which is alſo abſurd. (T. IX. B. I.)

COR. I. Hence any arch touches a circle only in one point.

COR. II. If an arch touch a circle in one point it is perpendicular to the diameter in that point.

COR. III. Hence if two circles touch one another, either inwardly or outwardly; the great circle paſſing through their centers, C, I, ſhall paſs through A, their point of contact.

Otherwiſe an arch touching both circles in that point could not be perpendicular to both diameters.

COR. IV. Any curve line A B, touching the circle A O only in the point A, will alſo be a tangent to all the other circles A V, whoſe centers C lie in the ſame great circle A I, paſſing through A and the center I of the circle A O.

COR. V. Hence, to draw a circle from a given center C, to touch another given circle O A, on the ſphere. Join their centers C, I, (P. I. B. I.) and produce C I (if neceſſary) to meet O A in A. Laſtly, with radius C A deſcribe the circle V A, and it is done.

PROBLEM IX.

Through a given point P on the ſphere, to deſcribe a circle P O of given curvature, which ſhall touch a given circle Q O.—(Pl. 2. fig. 2.)

Join P and the pole I of the circle Q O, (P. I. B. I.) and with radius P I and pole I deſcribe the circle V P. Produce P I, till A Q be equal to the radius of the required circle P O. About the pole A, with diſtance A Q deſcribe the circle Q V cutting V P in V. Alſo, about P, as a pole, with radius P O equal to V Q deſcribe a circle to cut Q O in O. Through P, O, (P. VI. B. I.) deſcribe the arch P O, having its center on that ſide of a great circle paſſing through P O, next to or from the point I, that the circle V Q hath with reſpect to the ſame point, and it is done.

Join I, V, (P. I. B. I.) becauſe I Q = I O, V I = I P, and V Q = P O (by hypotheſis) the triangles I V Q, I P O are equal and ſimilar, (T. V. B. I.) and the angle Q = O. And ſince V Q touches Q O at Q, (T. XVII. B. I) therefore P O touches Q O alſo at O.

COR. I. All arches V Q, O P, of equal curvature, each of which is convex or concave to the circumference V P, touching Q O in Q, O, and meeting V P in V and P reſpectively, will be equal.

COR. II. All equal arches V Q, O P, of equal curvature and convexity towards P V, are equally diſtant from I.

PROBLEM X.

With two circles on the ſphere A B, A C, to form an angle A B C equal to an angle F D E, formed by two circles D F, D E, of equal curvature, each to each.—(Pl. 2. fig. 4, 5.)

Through the poles Q, O, of the circles D F, D E, draw the arch Q O, cutting their circumferences in F and E reſpectively. On A C, lay off A C = D E, and draw B C perpendicular thereto, and equal to F E. Through A B, draw the arch A B of equal curvature with D F, (P. VI. B. I.) having its center V on the ſame ſide of A C that Q is of D E, and it is done.

The angles E and C are equal; (D. 30.) alſo, A C = D E and B C = F E, by conſtruction; therefore, angle A = D. (T. III. B. I.)

SCHOLIUM. The arch B C being always a determinate quantity when the curvature of the circles A C, A B, and angle of inclination are given, may, in all caſes, therefore be taken as the meaſure of that angle.

PROBLEM XI.

Through a given point P on the ſphere, to draw a circle A I P of given radius, forming a given angle P A V, with another given circle A V. — (Pl. 2. fig. 6.)

ANALYSIS. Suppoſe it done, and through Q and O, the reſpective poles of A V and A I P, draw the arch O Q (P. I. B. I.) meeting A V in V.

Becauſe the angle I A V is given, the arch V I muſt be given, (ſee Scholium P. X.) and hence alſo V Q − V I = I Q. Whence a circle deſcribed with I Q, as radius and pole Q will touch A I P in I. (T. XVII. B. I.)

Wherefore with radius Q I having deſcribed a circle I i; through P draw the circumference P I A touching I i in I, (P. IX. B. I.) and the thing is done.

COR. I. With radius Q O and pole Q deſcribe the circle O L, which will be the Locus of all the centers O, L, of the circles of equal radii with A I, and cutting A V in equal angles: For, join i, Q, (P. I. B. I.) and produce i, Q, to meet the circle O, L, in L: Then, is i L = i Q + Q L = I Q + Q O = I O; and, conſequently, the angle V a i (made by its circle a i with A V) = V A I.

THEOREM XVIII.

If A B, A V, be two circles on the ſphere given in magnitude and poſition; and B V a given circle, cutting the former circles in B and V reſpectively: Then, no other circle, b v of equal radius, having its center on the ſame ſide of the point A, can cut A B, A V, ſo as to make the angle A b v equal to

A B V, and A v b equal to A V B.—(Pl. 2. fig. 3.)

For, ſuppoſe the angle A b v (made by the circle b v of equal curvature with B V) =A B V; and from Q, the pole of the circle A V, draw the circle I i, (C. V. T. XVII. B. I.) touching B V in I: And, prolong Q I, (drawn through Q, i) (P. I. B. I) until it meet the pole C of B V. With radius Q C and pole Q, deſcribe the circle C c: Alſo, with center v and radius v c=I C, deſcribe an arch cutting the circle C c in c: And with the ſame radius and pole c, deſcribe the arch v i: Join v, c, and deſcribe L C, the locus of the centers of all the circles B V, b v, of equal curvature, cutting A B at equal angles A b v, A B V. (C. I. P. XI. B. I.)

Becauſe v c=I C, and the points C, c, lie in the ſame circumference; and alſo, becauſe C I touches I i in I; the circle V i will alſo touch the circle I i in i; (C. I. P. XI. B. I.) and, conſequently, angle A v i=A V I. Now, as the center of the circle v i is in the circumference C c, and not in the point C, (by ſuppoſition) and the center L, of the circle b V, lies in the circumference L C, and not in the point C (by ſuppoſition); the circles b v, v i, muſt cut each other at v; and, therefore, theſe circles are not in the ſame circumference; conſequently the angle A v b is unequal to A v i, or its equal A V B.

THEOREM XIX.

If two ſpherical triangles A B C, a b c, compoſed of arches of equal radii in each; that is, having the radius of the arch A B in the one, equal to the radius of the arch a b in the other, &c. have alſo all their angles reſpectively equal, the wholes are equal.—(Pl. 2. fig. 21, 22.)

If it be denied, let the angle b a c be applied to B A C, and let the point b fall on A B at b, and the point c on A C at C.

By ſuppoſition, angle A b c = A B C, and angle A c b = A C B; alſo the curvature of b c equal to B C. But this is impoſſible.—(T. XVIII. B. I.)

COR. I. If the ſides B A, A C, and b a, a c, be of equal curvature, forming each one continued arch B A C, and b a c; then will the lunar areas B A C, b a c, be alſo equal. —(Pl. 1. fig. 5, 6.)

THEOREM XX.

If a great circle V I C be perpendicular to another circle B C A at C, and there be taken B C equal to C A; then any two chords B I, I A, drawn to meet in V C, will be equal. —(Pl. 1. fig. 24.)

By C. II. P. V. B. I. the pole V of the circle B C A is in the arch V. Draw B V, V A, (P. I. B. I.) and alſo the chords B V, V A, and V I. Then, ſince B V = V C = V A, and B C = C A, the triangles B V C, C V A, are equal and ſimilar, (T. V. B I.) and the angle B V C = C V A. Alſo, ſince the chord B V = V A, and V I, common, and the angle B V I = C V I, (C. II. T. VII. B. I.) the triangles B V I, A V I, are equal; (4. I. E.) and, conſequently, B I = I A.

COR. I. Join B I, and A I, with circles of equal curvature, (P. VI. B. I.) then arch B I = A I.

COR. II. If I C be perpendicular to B C A, and arch B I be equal to A I in length and curvature; then arch B C = C A.

COR. III. If B I, I A, be equal arches of great circles, and B C = C A; then will the angles B I C, C I A, be equal.

For chord B I = I A, chord B C = C A, and chord I C common to the triangle B I C = C I A; therefore angle B I C = C I A. (C. III. T. VII.)

COR. IV. If B I = I A be arches of great circles, and angle B I C = A I C; then any circle B A paſſing through B C, will be biſected in C.

COR. V. If I C be perpendicular to B C A, and the angles B I C, A I C, equal; then will B C = A C, and B I = A I.

COR. VI. If B I = A I, and C I be perpendicular to B A; then will B C = A C, and angle B I C = A I C.

COR. VII. Suppoſe I the pole of the circle B C A, and draw any arch B L A; then if the angle B L I be equal to I L A, the parts B L, L A, will be equal and the contrary.

COR. VIII. If I C be perpendicular to B L A, and the chords B I, A I, equal; then is the arch B L = L A.

THEOREM XXI.

Three perpendicular arches drawn from the middle of the three ſides of any ſpherical triangle A B C, all meet in one point O.—(Pl. 2. fig. 7.)

Let E, F, be the middle points of A B and B C reſpectively, and E O, F O, two perpendiculars. From O draw O D perpendicular to A C; and join B, O; A, O; and C, O. (P. I. B. I.) By C. I. T. XX. B. I. A O = B O = C O; and, therefore, ſince D O is perpendicular to A C, A D = D C. (C. II. T. XX. B. I.)

COR. I. A circle deſcribed with radius C O and pole O, will paſs through A and C.

THEOREM XXII.

If a great circle B L b be drawn through the points B, b, of interſection of two equal circles B I B, B i b; and another great circle V i L I Q be drawn through their poles V, Q, then will B b and V Q mutually biſect each other in L.—(Pl. 2. fig. 8.)

Join V, B; B, Q; b, Q; and V, b. (P. I. B. I.)

The iſoſceles triangles V b Q, V B Q, having the angles at V and Q all equal, (T. V. B. I.) are alſo equal and ſimilar; and conſequently angle B V Q = b V Q. Whence B L = L b. (C. IV. T. XX. B. I.) And in the ſame manner it may be proved that V L = L Q.

COR. I. From V to any point P, in the circumference B P I, draw V P. (P. I. B. I.) And through O, where this meets B b and pole Q, draw Q O, (P. I. B. I.) meeting the circumference B R i in R. Then will V O = O Q, and P O = O R.

Becauſe V L = L Q, and O L common to the triangles V O L, Q O L; alſo the angles at L equal: Therefore the arch V O = O Q; (C. I. T. XX. B. I.) and hence alſo P O = V P − V O = R Q − O Q = R O.

COR. II. Any two perpendiculars P O, R O, let fall from a point O in B b, to the equal arches B I, B i, are equal.

COR. III. If two perpendiculars P O, R O, erected upon the equal circles B I, B i, and meeting in O, be equal; then is that point in the arch B b.

COR. IV. If two perpendiculars P O, R O, erected upon the equal circles B I, B i, and meeting in O be equal; then is the arch B R equal to the arch B P and the converſe.

For, ſince V O = O Q; V B = B Q and B O common, the angles B V O, B Q O, are equal: (C. V. T. VII. B. I.) Alſo, becauſe V B = B Q, and V P = R Q; the triangles V B P, Q B R, are equal and ſimilar; (T. III. B. I.) and, conſequently, B R = B P.

And, in like manner, it may be proved, that if the arch B R = B P; then will P O = R O.

COR. V. A circle deſcribed with center O and radius P O = R O, will touch both arches B R and B P in R and P reſpectively; and, converſely, any two arches B R, B P, of equal curvature, drawn from any point B to touch the ſame circle, will be equal.

THEOREM XXIII.

If the three ſides of any ſpherical triangle A B I, compoſed of circles of equal curvature, be produced till they meet on the oppoſite ſides, viz. the circles B I, A B, till they meet in b; the circles B I, A I, till they meet in i; and the circles A B, A I, till they meet in a: Then, the great circles B b, I i, and A a, drawn through the points of oppoſite biſection, will all meet in the ſame point.—(Pl. 2. fig. 11.)

Let O be the point where A a and B b meet, and from thence let fall the perpendiculars O V, O D, and O E, upon the three ſides A B, B I, and A I, reſpectively. (P. VIII. B. I.)

Since V O, O D, meet in B b, they are equal: (C. II. T. XXII. B. I.) Alſo, ſince O V, O E, meet in A a, they are alſo equal; (C. II. T. XXII. B. I.) and, therefore, D O = O V = O E: Whence the point O is alſo in the arch I i. (C. III. T. XXII. B. I.)

COR. I. Three arches biſecting the three angles of a ſpherical triangle compoſed of great circles, all meet in one point.

COR. II. Segment A E + the oppoſite ſide, B I = half the ſum of the ſides.

For half the ſum of the ſides is equal A E + D I + B D.

COR. III. Hence, to inſcribe a circle V D E in a given ſpherical triangle A B I of equal curvature, and having its ſides all concave or all convex to their reſpective oppoſite angles. Biſect any two angles B, A, with the arches B b, A a; and from their interſection O, let fall the perpendicular V O on any ſide A B, and it will be the required radius.

THEOREM XXIV.

Three arches drawn from the three angles of a ſpherical triangle (compoſed of great circles) to the middle of the oppoſite ſides, all meet in one point.— (Pl. 2. fig. 18.)

Let B D, A E, biſect the oppoſite ſides A C, C B, and through the point of interſection G, draw C G K meeting A B and K. (P. I. B. I.) Draw the chords A d C, C e B, and A k B; alſo, to O, the center of the ſphere, draw D O, E O, K O, and G O.

Becauſe A d C is in the ſame plane with D O, they will interſect in ſome point d; but the radius D d O, which biſects the arch A C, will alſo biſect its chord A d C; conſequently, A d = d C. In like manner we prove C e = e B.

Suppoſe, the plane triangle A C B, (Pl. 2. fig. 14.) taken out of the figure A C B O, and join B, d; A, e; and C, k.

Now, C k being in the plane C O K; (Pl. 2. fig. 18.) and, conſequently, paſſing through ſome point in G O, muſt cut both the planes D O B, A O E. But C k is alſo in the plane A C B; therefore, if it cut theſe planes, it muſt cut them in their point of interſection g, in the plane A C B.

By Emerſon's Geometry (P. XXXI. B. II.) A B is biſected in k: Hence, if the radius O k K biſect the chord A k B in k, it will biſect the arch A B in K.

LEMMA I.

Let A O D, A I D, be any great circles drawn through the poles A, D, of the great circle O I V; and let B E F C be any ſmall circle, having its centre O a quadrant diſtant from A and D, cutting A I D in and F, and A O D in B and C. Alſo join B, C, and E F, with ſtraight lines. Then will the chord E F be parallel to the chord B C.—(Pl. 2. fig. 12.)

Join O E and O, F, with great circles; (P. I. B. I.) and about A and D as poles, with the diſtance A B = D C, draw the parallel circles B b, C c cutting A I B in b and c reſpectively. Alſo join b, c, with a ſtraight line.

Becauſe the arch O E = O F, and O I common; alſo the angles at I both right, (P. VIII. B. I.) the arch E I = I F. (C. II. T. XX. B. I.) Alſo, ſince A I = D F, A E = D F. (A. 5.) Again, ſince A b = D c, it is plain that b c muſt be parallel to E F; but b c is equal and parallel to B C; conſequently E F is alſo parallel to B C. (9. XI. E.)

LEMMA II.

If a great circle A O D, paſſing through O, the pole of a leſs circle B A C D, be alſo perpendicular to another great circle B V C; then will the chords A D, B C, of theſe circles, drawn in the plane B A C D, cut each other at right angles in I, their point of interſection. —(Pl. 2. fig. 10.)

Join B, A, and A, C, with ſtraight lines. Becauſe B V C is perpendicular to A V D, the plane B I C V is alſo perpendicular to the plane D I A V; and ſince B V = V C, (C. VIII. T. XX. B. I.) conſequently B I = I C. Alſo ſince A I is common to the two triangles B A I, A I C, the angle B I A is equal to C I A.

COR. I. Draw any other great circle a d, paſſing through the poles of B V C, and conſequently cutting it at right angles in v; and join a, d, cutting B I C in i; then is a i perpendicular to B I C.

For a i is parallel to A I; (LEMMA I.) and, conſequently, perpendicular to B I C.

COR. II. Join v, i, and V, I; then will v i and V I produced, meet in the centre P of the ſphere.

For I V falls in the plane B V C, and alſo in the plane A V D, and conſequently in their common interſection which (C. VII. T. I. B. I.) paſſes through the center P. And in like manner it may be proved that i v paſſes through P.

THEOREM XXV.

Three arches drawn from the three angles of a great circle ſpherical triangle, perpendicular to the oppoſite ſides, all meet in one point.—(Pl. 2. fig. 17.)

Let B D, A E, be perpendicular to the oppoſite ſides A C, B C; and through the points of interſection G, draw C G K. Draw the chords A d c, c e B, and A k B; and let d, e, be the points where the planes d D B, e E A, of the circles D B, A E, interſect the planes D C A and B C E reſpectively. Draw the lines D d O, E e O, which (by C. I. and II. L. II.) will meet in O, the centre of the ſphere, and join K, O, and G, O.

Now, ſuppoſe the plane triangle A C B (Pl. 2. fig. 20.) taken out of the figure A C B O, and join B, d, A, e, and c, k; then will B d, A e be reſpectively perpendicular to A C, C B. (C. I. L. II.) And it may be proved exactly as in T. XXIV. B. I. that C k will paſs through their point of interſection g. Hence by Emerſon's geometry, (P. XXXIV. B. II.) C k is perpendicular to A B; and hence (by converſe C. I. L. II.) C K is perpendicular to A K B.

THEOREM XXVI.

Of two great circle ſpherical triangles A B C, A B G, having two ſides A B, B C, of the one, equal to two ſides A B, B G, of the other, each to each reſpectively; the baſe of that A B C will be the greateſt, which is ſubtended under the greateſt angle A B C.—(Pl. 2. fig. 19.)

Draw G C. (P. I. B. I) The angle B C G = B G C, (T. VIII. B. I.) is greater than A C G; and hence much more is A G C greater than A C G; therefore A C is greater than A G. (T. X. B. I.)

THE ELEMENTS OF SPHERICAL GEOMETRY.

BOOK II.

DEFINITIONS.

1. A Spherical quadrangle is a portion of the ſurface of the ſphere bounded by four arches of circles.

2. The diagonal of a ſpherical quadrangle is a great circle drawn between any two oppoſite corners.

3. A ſpherical parallelogram is a quadrangle, whoſe oppoſite ſides and angles are equal.

4. A ſpherical rectangle is a parallelogram, whoſe oppoſite ſides are equal, and whoſe diagonals are alſo equal.

5. A ſpherical rhombus is a parallelogram, whoſe ſides are equal, and angles oblique.

6. All ſpherical figures contained under more than four arches of circles, are called ſpherical polygons; thoſe having five ſides, are called ſpherical pentagons; thoſe having ſix ſides, are called ſpherical hexagons; and ſo on.

THEOREM I.

If two equal circles on the ſphere V B A, V D A, meeting in V and A, be interſected by two other equal circles B D, E C, making the oppoſite angles equal, B equal to C, and E to D; then will the ſide VD be equal to EA, and BV to CA.—(Pl. 2. fig. 15.)

Since angle VBD=ACE; BDV=AEC; and V=A: The triangles BVD, EAC, are equal and ſimilar; (T. XIX. B. I.) conſequently, VD=EA, and BV=CA.

COR. I. The figure BDCE is a ſpherical parallelogram.

For BD=EC; and, becauſe, VD+AC =EA+BV, if each of theſe be taken from the equals VDA, VBA: Then, DC=BE. (A. 5.) And ſince angle D=E, and B=C, the area BDCE is a parallelogram. (D. 3. B. II.)

COR. II. $DC=VA-\overline{AE+AC}$.

THEOREM II.

A great circle AD paſſing through the oppoſite corners of any ſpherical parallelogram ABDC, divides it into two equal and ſimilar triangles ABD, ACD. —(Pl. 2. fig. 16.)

For, ſince AB=CD, BD=AC, and

angle B=C. (D. III. B. II.) The triangles ABD, ACD, are equal and ſimilar. (T. III. B. I.)

THEOREM III.

The diagonals of a ſpherical parallelogram ABDC, interſect each other in the middle P of each AD and BC.—(Pl. 2. fig. 16.)

In the triangles APC, DPB, the angle CAP=PDP; (T. II. B. II.) and alſo, ACP =PBD; (Ibid.) whence, ſince CA=DB, the ſide AP=DB and CP=PB. (T. IV. B. I.)

THEOREM IV.

Any great circle BC paſſing through the middle of the diagonal of a ſpherical parallelogram AFDE, divides the area into two equal and ſimilar parts. —(Pl. 1. fig. 23.)

Draw the diagonal AD, which will interſect BC in the middle P:

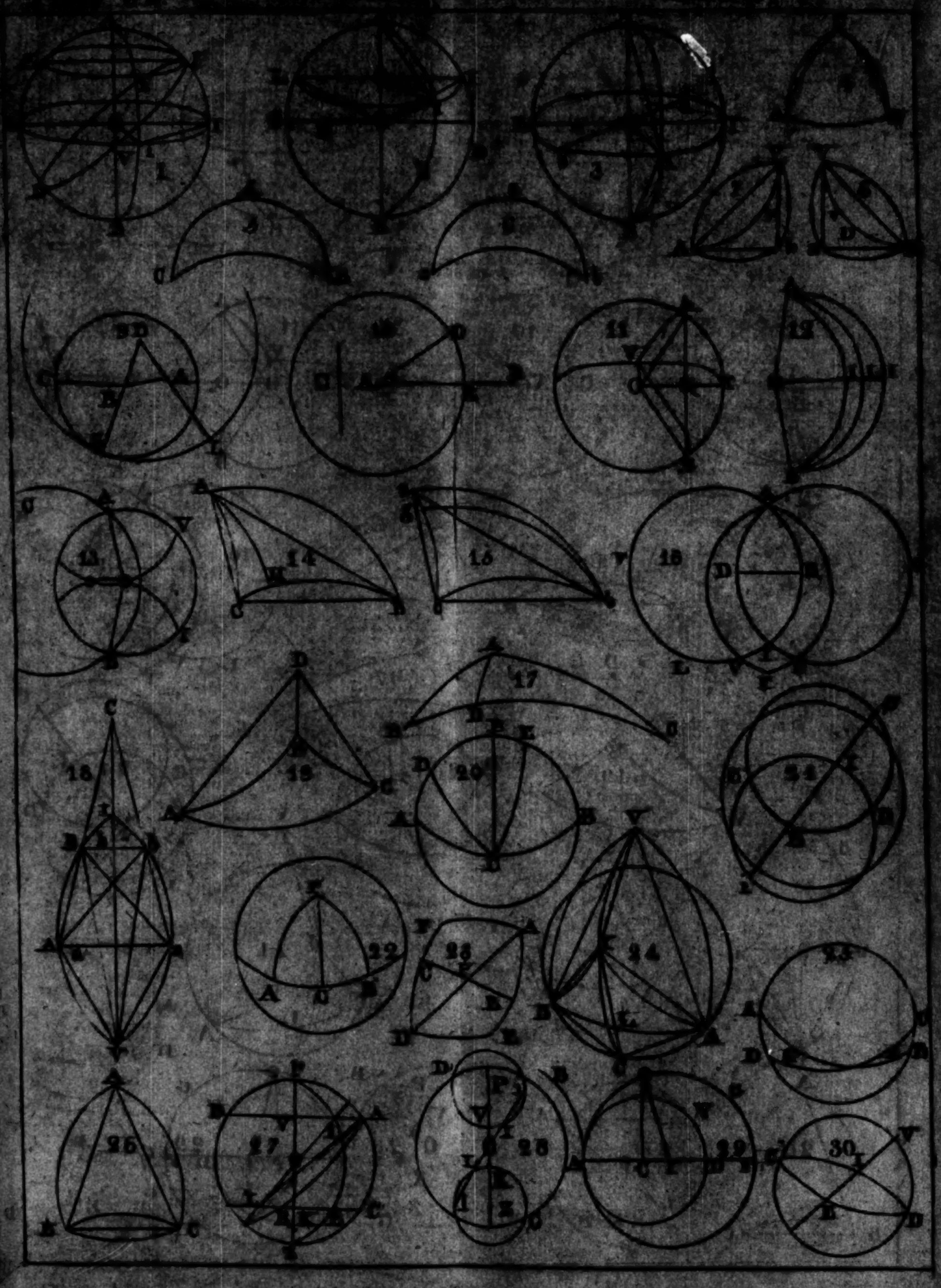
For, the triangles A P B, D P C, are equal and ſimilar; (T. IV. B. I.) and, conſequently, B P = P C. Take each of theſe triangles from the equal triangles A E D, A F D; (T. II. B. II.) and the remainders B P D E, C P A F, are equal; (A. 5.) therefore, E B C D = B P D E + P C D = C P A F + P B A = B A F C.

COR. I. Any great circle A P D paſſing through the middle point P, of the diagonal of a ſpherical parallelogram, is biſected in that point.

THEOREM V.

Spherical parallelograms A B C D, B C F E, whoſe ſides A B, D C, and B E, F C, are great circles; ſtanding upon the ſame baſe B C, and between equal and parallel circles A F, B C, are equal. —(Pl. 3. fig. 1.)

For, ſince the angle F = B E A and A = C D F; (C. II. T. XV. B. I.) alſo, A B = D C; (D. 3. B. II.) the triangles A E B, D C F, are equal and ſimilar: (T. XVI. B. I.) Therefore, the triangles A E B, D C F, each



taken from the whole figure A B C F, will leave A B C D = E B C F.

COR. I. Hence, ſpherical triangles B A C, B F C, whoſe ſides B A, A C, and B F, F C, including the vertical angles A, F, are great circles, ſtanding upon the ſame baſe B C, and between equal parallel circles A F, B C, are alſo equal.

For they are the halves of their reſpective parallelograms. (T. II. B. II.)

COR. II. Through B, C, draw the great circle B V C: (P. I. B. I.) Then is the triangle B A C V = triangle B F C V.

For the ſpace B C V is common to the equal triangles B A C, B F C.

PROBLEM I.

Through three given points A, I, P, on the ſphere, to draw two equal and parallel circles A a and I P. — (Pl. 3. fig. 2.)

Through the points A, I, draw the great circle I A a i: (P. I. B. I.) And from E, the middle of A I, lay off E b = E B = a quadrant; (P. III. B. I.) alſo, from the points B, b, lay off upon I A a i, B A = B a = b i = b I reſpectively. Through the points I, P, i, draw a circle; (5. IV. E.) and with the ſame radius, through the points A, a, draw another circle, (P. VI. B. I.) and it is done.

Draw A V a, I v i, and alſo B b, paſſing through C, the center of the ſphere; alſo, to O, Q, the reſpective centers of the circles A a and I P, draw C O and C Q: And through O V and Q v, draw the lines O V, Q v, reſpectively; alſo, join O, A; I, Q.

Becauſe A B = a B; the chord A a is biſected in V; and, conſequently, A V = a V: Alſo, the angles at V, in the plane b A B a, are all right ones:

And in the ſame manner it may be proved, that v I = i v; and the angles at v, in the ſame plane, are right ones.

Again, ſince A B = a B = i b = I b; the lines A V, a V, i b, and I b, will alſo be equal; and V B = v b. Whence, conſequently, V C = v C: Again, ſince a V A is biſected by V O, which paſſes through the center O, of the circle O A a; the angle O V A is a right angle: (3. III. E.) and, in like manner, it is proved, that I v Q is alſo a right angle.

Alſo, ſince O A = I Q; (by conſtruction) the triangles O V A, Q v I, are equal and ſimilar: Whence O V = Q v.

Again, ſince C O is perpendicular to the plane O A a; (C. IV. T. I. B. I.) and C Q alſo perpendicular to the plane I P i Q; the angles C O V, C Q V, are each right ones; and the triangles C O V, C Q v, equal and ſimilar: Hence, angle O C V = Q C v and O C = C Q.

Alſo, ſince the angles O V C, Q v C, are equal, and V A, V I, both perpendicular to V v: And the angles O V A, I v Q, each equal a right angle; the ſections A a, I i, are parallel; alſo, ſince O V is perpendicular to a A, and v Q to i I, the angles O V C, C v Q, which the planes a A O, Q I i, form with the plane a A I i, are equal, and theſe planes, of conſequence, will be parallel; and the circles a A O, i I P, are equal by conſtruction.

THEOREM VI.

From the equal ſides A B, A C, of a great circle Iſoſceles triangle A B C, let

there be laid off A D equal to A E, and join the oppoſite points B, E, D, C, with great circles; (P. I. B. I.) then will the areas of the triangles A D C, A E B, be equal, and the oppoſite angle A D C be equal to A E B, and A B E to A C D. —(Pl. 2. fig. 9.)

Through B, C, and E, draw the equal and parallel circles B C and D E, (P. I. B. II.) which, becauſe the points D, E, are equidiſtant from the plane of the leſs circle B C, B E, will alſo paſs through D.

By (C. II. T. V. B. II.) the triangles D B C, E B C, are equal. Take each of theſe from the whole triangle B A C, and the remainders A D C, A E C, will be equal. (A. 5.)

Again, the angles A D C, A E B, are evidently equal. (C. V. T. VII. B. I.) And in the ſame manner it will be proved that A B E = A C D.

COR. I. If two triangles A D C, A E B, compoſed of great circles, have their vertical angles A equal; and alſo the oppoſite angle A D C = A E B, &c. the areas will alſo be equal.

COR. II. If the plane triangles B D C, C E B, compoſed of the chords of the ſpherical arches of the triangles B D C, C E B, be equal and ſimilar; the ſpherical triangles will alſo be equal in area.

THEOREM VII.

Every great circle ſpherical triangle A B C, hath (by producing its ſides,) another triangle a b c on the oppoſite ſide of the ſphere equal to it in area.—(Pl. 3. fig. 13.)

The point a is oppoſite to A, or a ſemicircle diſtant from it; and b is oppoſite to B, and c to C. Likewiſe angle a = A, b = B, and c = C. Therefore the triangle a b c is equal A B C. (C. I. T. VI. B. II.)

COR. I. Hence any great circle ſpherical triangle A B C, by producing its ſides, hath oppoſite to every angle thereof, another triangle of the ſame baſe, and oppoſite angle with the former; and the other parts the ſupplements thereof.

Thus in the triangle B c a, angle B = A B C, a c = A C, a B = ſupplement A B, c B = ſupplement C B, angle c a B = ſupplement C A B or A, and angle a c B = ſupplement C.

COR. II. Any great circle ſpherical triangle A B C, has adjoining to every ſide thereof another triangle, having the ſame ſide and oppoſite angle, and the other parts the ſupplements thereof.

Thus in the triangle A B c, A B, and angle c, are the ſame with A B and angle C; B c, A c, are ſupplements of B C, A C; angle c B A = ſupplement C B A, and angle c A B = ſupplement C A B. Likewiſe there is the triangle a B C adjoining to C B; and the triangle A b C adjoining to A C.

THEOREM VIII.

The area of any great circle ſpherical triangle A B C, is equal to the radius of the ſphere into the ſum of the meaſure of the angles A, B, and C, minus one fourth the ſurface of the whole ſphere. —(Pl. 3. fig. 14.)

For having produced all the ſides of the triangle A B C till they interſect again, and form the equal triangle a b c. (T. VII. B. II.) Call each of theſe P; put Q = triangle A C a, T = triangle B C b, and R = a C b; (where P + Q

$+T+R$ are ſuppoſed to be included in one hemiſphere); put $\frac{S}{2} = \frac{1}{2}$, the ſurface of the ſphere; and put A, B, C, for the meaſure of theſe angles reſpectively; alſo, put $R =$ radius of the ſphere. Then, by Dr. Hutton's Menſuration of Solids, P. 20. 1. edition, we have $2\ R \times$ angle $A = P + T$, $2\ R \times$ angle $B = P + Q$; and $2\ R \times$ angle $C = P + R$: Hence, $2\ R \times \overline{A+B+C} = 3\ P + Q + R + T = 2\ P + \frac{S}{2}$; conſequently, $P = R \times \overline{A+B+C} - \frac{S}{4}$

THEOREM IX.

If two great circle ſpherical triangles ABC, DEF, have the ſum of their angles equal, they will be equal in area and converſely, if they are equal in area the ſum of their angles will be equal. —(Pl. 5. fig. 1. 2.)

For retaining the notation in the laſt Theorem, we have $A+B+C = \frac{P + \frac{1}{4}S}{R}$: And ſuppoſing Q equal to the area DEF: Then

$D+E+F=\frac{Q+\frac{1}{4}S}{R}$, conſequently $\frac{P+\frac{1}{4}S}{R}=\frac{Q+\frac{1}{4}S}{R}$, and $P=Q$.

Again, ſince $R.\overline{A+B+C}-\frac{1}{4}S=P=\overline{D+E+F}.R-\frac{1}{4}S$, we have by reduction $A+B+C=D+E+F$.

SCHOLIUM. From this theorem naturally flows the determination of ſolid angles: For if we ſuppoſe O and V the reſpective centers of the ſpheres D O E F, A V B C, and O Q, V P, any two elementary pyramids of equal and ſimilar baſes Q and P, it is plain their ſolidity will be equal: Alſo, ſince the triangles D F E, A C B, contain an equal number of the baſes Q, P; the ſolids, or ſolid angles D O E F, A V B C, muſt conſequently be alſo equal: And ſince the ſum of the angles in each are in this caſe alſo equal, their meaſure will aptly determine the ſolid angles. Hence, ſuppoſing the arches A C, C B, B A, produced each way (if neceſſary) till they form quadrants A c, C b, B I, and that A L be taken equal to A c, C a = C b, B G = B I: Then, joining c, L, a, b, and I, G, we have c L + a b + I G, for the meaſure of the ſolid angle A V C B.

COR. I. Hence alſo, if the angles A, B, C, be ſeverally given, the ſolid angle A V C B, is given in ſpecies.

THEOREM X.

If two ſpherical triangles C A B, I V L, have the ſide C A in one equal, the ſide I L in the other; the ſide A B equal to I V, and the ſide C B equal to V L: Alſo, each pair being of equal curvature, and both concave, or both convex to their oppoſite angles; then will the areas be equal.—(Pl. 4. fig. 11. 12.)

Draw the great circles C a A, A b B, B c C, I i L, L l V, and V v I; (By C. II. T. VI. B. II.) the triangle C a A b B c = I i L l V v; and alſo, the lunar ſpace C A a = I L i; and in like manner, C B c = V L i and A B b = I V v: Hence, taking the three equal lunar areas in each, from their reſpective triangles; there will remain the ſpherical triangle C A B = I V L.

COR. I. If the plane triangles C A B, V I L, compoſed of the chords of the ſpherical triangles C A B, V I L, be equal and ſimilar, the ſpherical triangles will alſo be equal in area.

For the propoſition is true of the great circle ſpherical triangles C a A b B c, V v I i L l, and muſt be alſo true when the equal lunar areas C a A, I i L, &c. are reſpectively added to or ſubtracted therefrom.

COR. II. If the plane angle C A B = V I L, the chord B A = V I, and A C = I L; (every thing elſe being ſuppoſed as in the propoſition) then will the ſpherical triangles C A B, V I L, be equal in area.

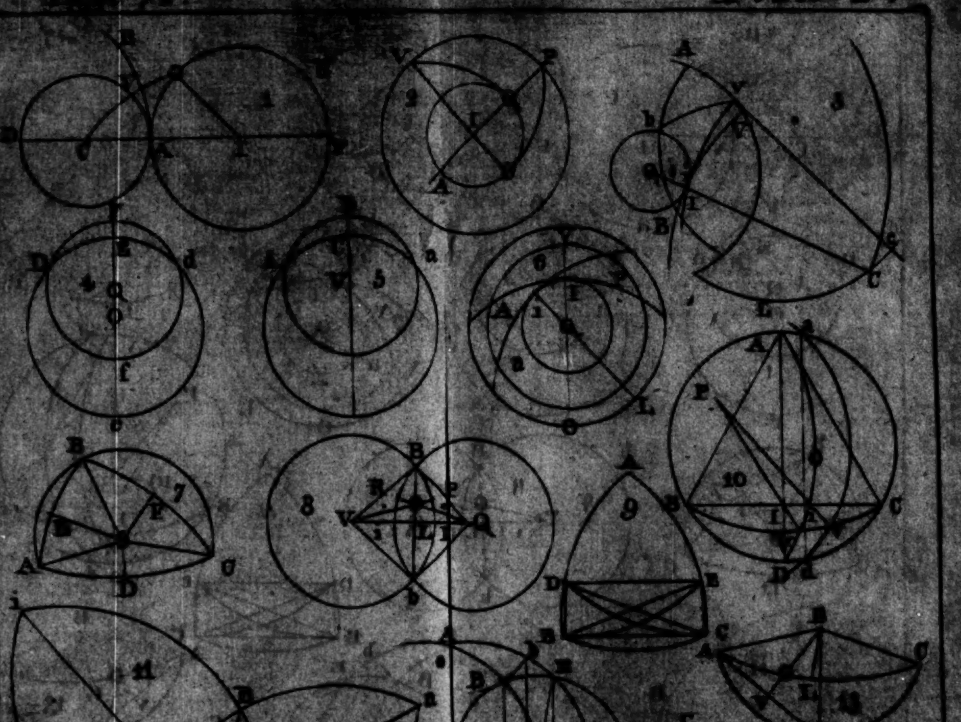
COR. III. If the arch B A = V I and A C = I L; alſo, B C, V L, of equal and ſimilar curvature, and angle A = I; then will the ſpherical triangles C A B, V I L, be alſo equal in area.

For the plane triangles C A B, V I L, are equal and ſimilar; (C. IV. T. VI. B. I.) and the areas C A B, V I L, are equal. (C. I. T. X. B. II.)

COR. IV. If the arch A C = I L, angle A = I, angle C = L, and the whole triangles C A B, V I L, are of equal curvature; then will the ſide B A = V I, B C = V L, and angle B = V.

This is plain from the converſe of propoſition, and C. IV. T. VI. B. I.

Plate 2.



THE

ELEMENTS

OF

SPHERICAL GEOMETRY.

BOOK III.

DEFINITIONS.

1. A Spherical angle D O B at the centre, is that whoſe angular point O is in the center of the circle D A B. Pl. 3. fig. 4.)

2. A ſpherical angle D A B at the circumference, is when the angular point A is in the circumference D B.—(Pl. 3. fig. 4.)

THEOREM I.

If from a point G on the ſphere, out

of the center of the circle A B, ſeveral arches G D, G E, &c. be drawn, the greateſt is that G F which paſſes through the center C; and thoſe nearer to G F are greater than thoſe further off.—Alſo G H, the remainder to G F, is the leaſt; and thoſe nearer to it, as G A, are leſs than thoſe further off, as G B.—(Pl. 3. fig. 3.)

Draw C E, C D, C A, C B; (P. I. B. I.) then G C + C E, or G F, is greater than G E. (T. IX. B. I.) Alſo, in the triangles G C E, G C D; G C, E C, are equal to G C, D C; but angle E C G is greater than D C G; conſequently, E G is greater than D G. (T. XXVI. B. I.)

Alſo C G + D G is greater than C D, or its equal C H. (T. IX. B. I.) Take away C G, and D G is greater than G H. After the ſame manner G A is greater than G H, and G B than G A.

COR. I. Only two arches drawn from G to the circumference, can be equal; and they muſt alſo lie on different ſides of the arch H C F.

For no two arches on the ſame ſide can be equal.

COR. II. If from any point C, three equal arches can be drawn to the circumference, that point is the pole or center of the circle.

COR. III. The chord of G F is greater than that of G E, and that of G E than that of G D. Alſo the chord of G H is leſs than tha of G A, and that of G A than that of G B.

COR. IV. Hence, if arches of equal curvature (the greateſt of which muſt not exceed a ſemicircle) be drawn through the points G, F, G, E, G, B; the greateſt will be that G F paſſing through F, the extremity of the diameter H G C F that is drawn through G, and the pole C of the circle A E B.

Alſo, thoſe that are nearer G F, as G E will be greater than thoſe, as G D, that are further off.

And hence alſo the arch paſſing through H B will be leſs than that paſſing through A G, &c.

THEOREM II.

If from a point G on the ſphere without a circle, ſeveral arches be drawn to cut it; of thoſe that paſs to the concave part, the greateſt is that G F which paſſes through the center; and thoſe nearer to G F are greater than thoſe that are further off.—But of thoſe that go to the convex part the leaſt, is that G H, which continued would paſs through the center C; and thoſe nearer to that, as G A, are leſs than thoſe that are further off.—(Pl. 3. fig. 12.)

Draw C A, C D, C B, C E; (P. I. B. I.) then in the triangle G C E, G C + C E, or G F is greater than G E. (T. IX. B. I.) And in the triangles G C B, G C E; G C, E C, are equal G C, C B, and angle G C E greater than G C B; conſequently G E is greater than G B. (T. XXVI. B. I.)

Alſo in the triangle G C A, C A + A G is greater than C G, or C H + H G. (T. IX. B. I.) Take away C A = H C, and A G is greater than H G. And in the triangles C A G, C D G; C G, C A, are equal to C G, C D, and angle G C A leſs than G C D; therefore G A is leſs than G D. (T. XXVI. B. I.)

COR. I. There can only two arches be drawn of equal length from the point G to the circumference of the circle; and they muſt alſo lie on different ſides of the ſame diameter. For no two are equal on the ſame ſide of G F.

COR. II. The greateſt arch to the convex part, or the leaſt to the concave part, is the tangent to the circle.

THEOREM III.

If a great circle ſpherical trapezium D A B O have one of its angles O at the center; and the others, D, A, B, at the circumference of any circle A B. The middle angle D A B at the circumference, ſhall be equal to both the others O D A + O B A. — (Pl. 3. fig. 4.)

Join A, O: (P. I. B. I.) Then, (T. VIII. B. I.) angle D A O = O D A and O A B = O B A: Hence, adding the latter equals to the former, we have D A O + O A B = A = O D A + O B A. (A. 4.)

PROBLEM I.

To make a ſpherical triangle D F E

of three given arches of circles A, B, C, of which the ſum of the chords of any two, muſt be greater than the chord of the third ſide.—(Pl. 3. fig. 18.)

Draw DE equal to A; (P. III. B. I.) then, with radius C and center E, deſcribe an occult arch Bf: Alſo, with radius F and center D, croſs the former arch at F; draw DF, EF, (P. I. B. I.) and DEF is the triangle required.

(By conſtruction) DE = A, EF = C, and DF = B. The reſtriction is evident from (22. I. E.)

COR. I. After the ſame manner a ſpherical triangle is made equal to a given ſpherical triangle.

THEOREM IV.

If a great circle ſpherical trapezium DABC be inſcribed in a circle, the ſum of its oppoſite angles A and C, are equal to the ſum of its other two oppoſite angles D and B.—(Pl. 3. fig. 4. 7.)

Join the center O of the circle, and the points D, B, and C. (P. I. B. I.)

CASE I. When the center of the circle O falls within the trapezium. (Pl. 3. fig. 4.)

The angle A = O D A + O B A, and angle C = C D O + C B O; (T. III. B. III.) conſequently, A + C = O D A + C D O + O B A + C B O (A. 4.) = D + B.

CASE II. When the center of the circle O falls without the trapezium. (Pl. 3. fig. 7.)

The angle A = O D A + O B A, and angle C = C D O − C B O (B C O); conſequently, A + C = O D A + C D O + O B A − C B O (A. 4.) = D + B.

COR. I. If the ſum of the oppoſite angles A + C are equal D + B. That great circle ſpherical trapezium is inſcribed in a circle.

COR. II. The ſpherical angle D A B (Pl. 3. fig. 9.) inſcribed in a ſemicircle, is an angle of a great circle rectangle.

For, make the triangle D C B equal and ſimilar to the triangle D A B, having the angle C = A, (P. I. B. III.) and ſide D C = A B:

Then, will C be in the circumference of the circle D A B, and angle A = C = D = B. (T. IV. B. III.)

PROBLEM II.

AT, At, are two equal arches given in magnitude and poſition, it is required to draw a circle to touch them in T and t reſpectively. — (Pl. 3. fig. 8.)

Erect the perpendiculars T O, t O, and from O, their interſection with radius T O, deſcribe the circle T P t, and the thing is done.

The arch T O = t O; (C. IV. T. XXII. B. I.) and T O touches A T in T, by conſtruction; alſo, (C. V. T. XXII. B. I.) it touches A t in t.

THEOREM V.

Let A B, A C, be two circles of equal curvature on the ſphere, touching a given circle T P t in T and t reſpectively, and B C a third circle of equal curvature with A B and A C likewiſe touching the circle in P, intercepted between AC and AB. Alſo let all theſe circles be either concave or convex to the circumference TPt; then will $AC + CB + BA$, be a conſtant quantity when CB is on that ſide of the circle next to A; and $Ac + Ab - bc$, will alſo be a conſtant quantity when bc is drawn in the ſide oppoſite to A.—(Pl. 3. fig. 8.)

For arch $TC = CP$, $PB = Bt$, $Tc = cp$, and $bp = bt$. (C. IV. T. XXII. B. I.) Therefore $AT = AC + CP$, $At = AB + BP$; conſequently $2AT = AT + At = AC + CB + BA$.

Again, $AT = Ac - Tc = Ac - pc$. (C. IV. T. XXII. B. I.) Alſo, $At = Ab - tb = Ab - pb$. Hence, $2AT = AT + At = Ac + Ab - bc$.

COR. I. If M, N, be two points taken in AC and AB reſpectively, on that part of TAt next angle A; then will the perimeter of the trapezium $CMNB$ be a conſtant quantity: And alſo, the difference between the three ſides $Mc + MN + Nb$, and the fourth ſide bc, of the trapezium $cMNb$, be alſo a conſtant quantity.

COR. II. The ſum of the two oppoſite ſides of the quadilateral ſpherical figure $cCBb$, circumſcribing the circle TPt, are equal to the ſum of the other two oppoſite ſides.

THEOREM VI.

If AB, AG, be two arches of equal curvature, both concave or convex to a given circle PT, and touching it in P, P, reſpectively: Then, if from the point A, we lay off AD equal to AD; alſo, from D, D, erect perpendiculars to AB, AG, meeting in C; alſo, with radius DC and center C, deſcribe the ſmall circle DtD. (P. II. B. III.) I ſay a third arch BG of equal curvature and concavity or convexity with AB, AG, and touching the two circles in T and t reſpectively, will interſect AB and AG in B and G, ſo that BG will be equal to PD.—(Pl. 4. fig. 1. 2.)

For $AB+BG+AG=2AP$, (T. V. B. III.) $AB+AG-BG=2AD$: (Ibid.) Whence, by ſubtraction, $2BG=2AP-2AD=2DP$; conſequently, $BG=DP$.

COR. I. The diſtance BT = t G.

Since BD = B t and PD = BG, by ſubtraction, BT = BP = PD − BD = BG − Bt = t G.

COR. II. The arch T t is equal to the difference of the ſides AB, AG, in the ſpherical triangle ABG.

For AB − AG = BD − DG = B t − t G = BG − 2 t G = T t. (C. I. T. VI. B. III.)

COR. III. On BG (produced, if neceſſary) let fall the perpendicular AI; and ſuppoſe BA + BI a given quantity; then will TI alſo be given.

Upon AB produced, lay off AV = AB + BI; alſo, from each of theſe equals take the equals AB + BP = AB + BT; then BV = TI.

THEOREM VII.

If from a point P, within the ſphere, out of the center C, ſeveral ſtraight lines AP, VP, IP, be drawn to the circumference; the greateſt is that PA, which paſſes through the center C, and thoſe as IP nearer to PA, are greater than thoſe, as VP, that are further off;

Alſo, a P (the remainder to A P) is the leaſt, and thoſe nearer to it, as P i, are leſs than thoſe, P v, that are further off. —(Pl. 3. fig. 5.)

Through V, A, deſcribe a great circle which, becauſe the points V, A, v, a, are all in the ſame plane, will paſs through V, a; alſo, through A I, deſcribe a great circle (P. I. B. I.) which will alſo paſs through a i.

By 7. III. E, the line A P is greater than P I: Alſo, ſince A I, V A, are arches of equal circles paſſing through their common diameter a A, whereof the point I is nearer A than the point V; therefore, the line P I is greater than P V. (7. III. E.)

Again, a P is leſs than P i: (7. III. E.) Alſo, it will be proved as before, that P i is leſs than P v.

COR. I. If from any point C, three equal lines can be drawn to the ſurface of the ſphere, that point is the center.

THEOREM VIII.

If from a point P, without the ſurface of the ſphere, ſeveral right lines be drawn to cut it. Of thoſe which paſs to the concave ſuperficies, the greateſt is that PF which paſſes through the centre C of the ſphere, and thoſe nearer to PF, as PI, are greater than thoſe, as PV, which are further of. But of thoſe that fall on the convex ſuperficies, the leaſt is that Pf, which, continued, would paſs through the centre C; and thoſe nearer to that, as Pi, are leſs than thoſe, as PV, that are further off. —(Pl. 3. fig. 19.)

Through V, F, deſcribe a great circle; (P. I. B. I.) which, becauſe the points V, F, f, v, are all in the ſame plane, will paſs through f, v. Alſo through F, I, deſcribe a great circle, (P. I. B. I.) which will alſo paſs through f, i.

The line PF is greater that PI. (8. III. E.) Alſo ſince FI, FV, are arches of equal circles paſſing through their common diameter Ff, whereof the point I is nearer F than the point V; therefore the line PI is greater than PV. (8. III. E.)

Again, Pf is leſs than Pi: (8. III. E.) Alſo it will be proved as before that Pi is leſs than PV.

COR. I. The greateſt line to the convex part, and the leaſt to the concave part, is the tangent to the ſphere.

THEOREM IX.

All the tangents are equal which are drawn from a given point P, without the ſphere T A, to the ſurface of the ſphere quite round—(Pl. 3. fig. 15.)

For let P T be any tangent from the given point P; and draw P C to the centre C, and join T, C. Alſo let C T A be a great circle of the ſphere in the plane of the triangle P T C; then C P and C T, as well as the angle T which is right, being conſtant in every poſition of the tangent, the ſquare of P T is conſtantly equal $PC^2 - TC^2$; and therefore as well as P T, is a conſtant quantity.

COR. I. If P T be a tangent drawn from P and T, C, joined T C will be perpendicular to P T.

COR. II. If tangents P T, P A, P t, be drawn to the ſurface of the ſphere quite round, the figure T P A t will be a right cone.

COR. III. Hence, if a ſphere be placed within a right cone ſo as to touch it in two points, then ſhall the outſide of the ſphere, and the inſide of the cone, mutually touch quite round, and the line of contact be a circle.

THEOREM X.

If from a given point P, a right line P I B be drawn in any poſition through a ſphere T I B, cutting its ſurface in two points I B, the rectangle contained under the whole line P B, and the external part P I will always be equal a conſtant magnitude, viz. the ſquare of the tangent P T drawn from the ſame point.—(Pl. 3. fig. 15.)

Let T, and the points I, B, be in the ſame great circle T I B; then (36. III. E.) $PI.BP = PT^2$ = a conſtant quantity. (T. IX. B. III.)

THEOREM XI.

If two lines A B, D F, interſect each other in any point P, within a ſphere

ADB, and be terminated at the ſurface ADBF, the rectangle AB, PB, of the parts of the one, will be always equal to the rectangle of the parts DP, PF, of the other. And univerſally the rectangles of the two parts of all lines paſſing through P are equal.—(Pl. 3. fig. 6.)

Conceive a plane to be drawn through AB, and the centre C of the ſphere, forming the great circle ABD; and draw its diameter DCPF through P, the point of interſection of all the lines; then the rectangle AP. PB = DP. PF. (35. III. E.)

Again, through any of the other interſecting lines GH and the center C, conceive another plane to paſs, forming the great circle DGFH. Then becauſe the points C, P, are in this latter plane, the line CP, and conſequently the whole diameter DCPF are in the ſame plane; and therefore DCF is a diameter of the circle DGF, of which GPH is a chord. Therefore again, GP. PH = AP. RD.

THEOREM XII.

If through the centers O, V, of two

unequal circles of the ſphere, a great circle be deſcribed, meeting their circumferences in I, i, and Q, q, reſpectively: And to the extremes I, Q; i, q, lines be drawn; theſe will meet in a point A; ſo that the cones I A i and Q A q ſo formed, by drawing lines from A to every point in each circumference, will be ſimilar.—(Pl. 3. fig. 10. 11.)

Becauſe the points I, i, q, Q, are in the ſame plane of the great circle I Q V O; the lines I Q; i q, will interſect each other in ſome point A: And ſince the points I, i, q, Q, are all in the ſame circle we have I A to i A as A q to A Q: (C. 36. III. E.) Alſo, ſince the planes of the circles I i and Q q are perpendicular to the plane of the great circle O V; (converſe of C. III. T. I. B. I. and D. 41. B. I.) the ſections made by them will, (by conics) cut the ſphere in ſub-contra directions; and, conſequently, the cones I A i and q A Q are ſimilar.

COR. I. Of the two common ſections of a ſphere and cone, if the one be a circle, the other will be a circle alſo.

THEOREM XIII.

If from any given point A within or without a given ſphere B F I, any right line A I be drawn, cutting the ſphere in I, and produced to V, ſo that A I may be to A V in a given ratio. The point V will always fall in the ſurface of a ſphere given in poſition.—(Pl. 3. fig. 20. 21.)

Let F B be any great circle of the ſphere, and through the center D draw the indefinite right line A D F G, and from A to any point B, in F B, draw the right line A B; and thereon produced, take A C to A B as A G to A F, the given ratio; then will the points G, C, fall in the circumference of a circle given in poſition. (Apollonius on plane loci.)

Again, let F I be any other great circle and through A, I, draw A I V, and take V A to I A as A G to A F. Now, the points, A I V G are in the ſame plane; and (by Apollonius on plane loci) the point V, in that plane, falls in a circle, V G, given in poſition; and hence, ſince the circle F I = F B, 'tis plain the circle G V = G C; conſequently, the points G, C, V, are all in the ſurface of the ſame ſphere.

THEOREM XIV.

Let G a A be any circle, and through I, its center, draw A I G F, ſo that F A G may be equal to a given rectangle; and through F draw D B F E at right angles to I F a: Alſo, through I parallel thereto draw O I Q; and ſuppoſe the plane D O Q E, by its revolution round O Q, to deſcribe the cylinder D L H K E: Then, if any line a i c b be drawn from the circumference G a, paſſing through the axis Q O in i, and touching the ſurface of the cylinder in b: Alſo, thereon be taken a c. a b = F A G, I ſay the point c will fall in the ſurface of a ſphere, of which G a A is a great circle; and, if the line F A be given, the cylinder D L H K E, and circle G a A, will be both given in magnitude and poſition.—(Pl. 4. fig. 16. 17.)

Through A, i, draw A i C B meeting F B

in B: Alſo, through a, I, draw a I g f meeting the circle A a in g, and the cylinder in f, and join b, f. Take $BAC = FAG$, then will the point C fall in the circumference of a circle G C A, given in magnitude and poſition, (Apollonius on plane loci.) Again, becauſe $Ia = IA$, and I i common to the triangles I i a, I i A, and the angles at I both right, the angles f a b, F A B are equal; Alſo, ſince $fa = FA$ we have $ab = AB$, and ſince $ac . ba = FAG = AC . AB$; conſequently, $ac = AC$; therefore, the point C falls in the circumference of a circle $gca = GCA$, a great circle of the ſphere G c A.

THEOREM XV.

If from a given point A, within or without a given ſphere, the lines A B, A C, and A b, A c, are drawn in the ſame ſtraight lines, meeting the ſurface of the given ſphere in B and b reſpectively, and making the rectangles $BAC = bAc$ equal to a given one; the points C, c, will alſo fall in the ſurface of another ſphere, given in magnitude and poſition.—(Pl. 3. fig. 16. 17.)

Let DE be a diameter of any great circle DFE of the given ſphere paſſing through the given point A, and let AB meet the circumference again in F; then (by Apollonius in plane loci) the point C falls in the circumference of a circle given in magnitude and poſition.

Again, draw any other great circle DfEb meeting bAc in f; then, (by ibid.) the point c falls alſo in the circumference of a circle given in magnitude and poſition: Wherefore, ſince the circle DFE=DfE, and the rectangle BAC=bAc; the points C, c, fall in the circumference of equal circles.

THE

ELEMENTS

OF

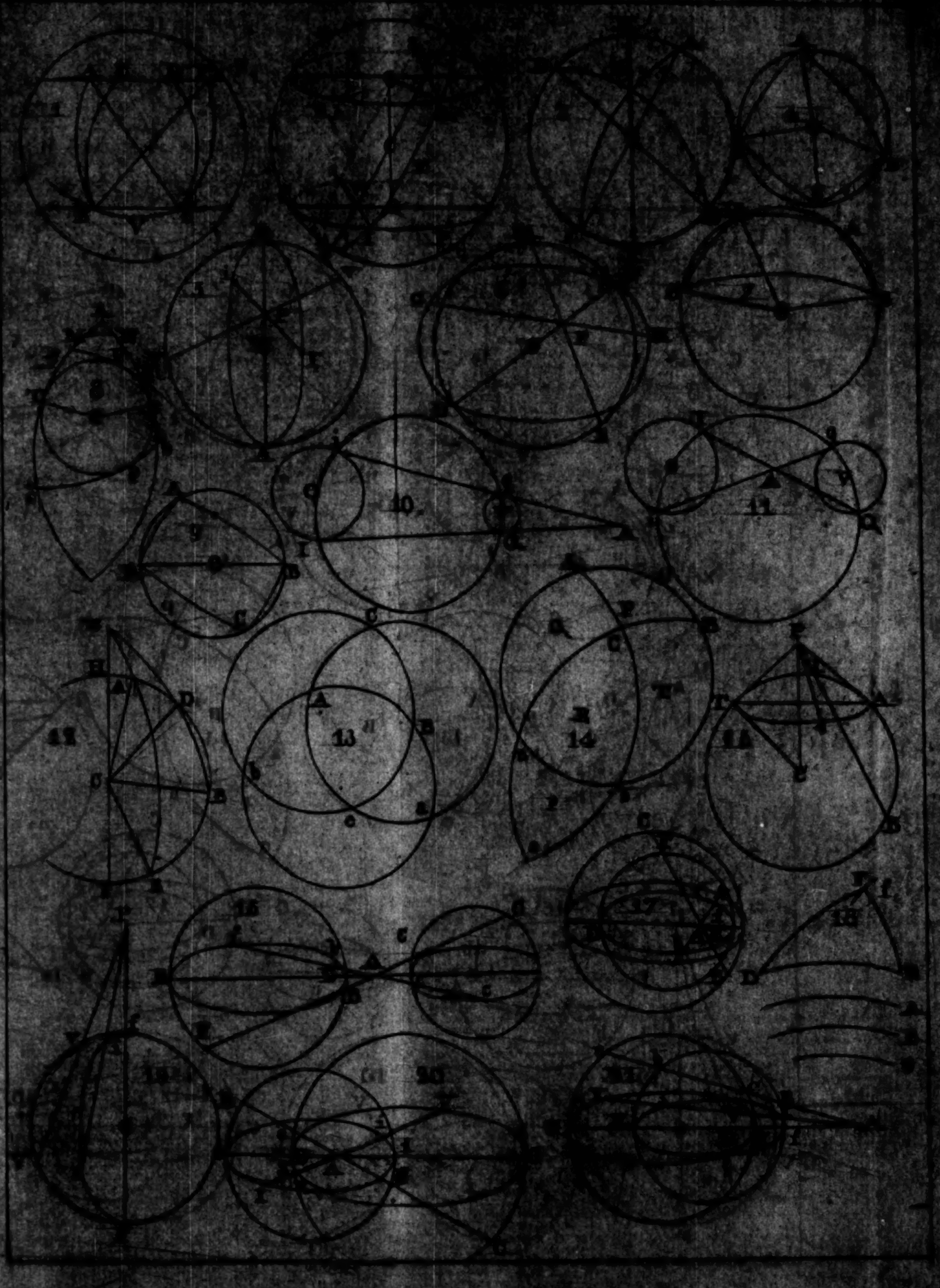
SPHERICAL GEOMETRY.

BOOK IV.

DEFINITIONS.

1. A Figure is ſaid to be inſcribed in a ſpherical circle, or a circle of the ſphere circumſcribed about a figure, when all the angular points of the figure are in the circumference of the circle, and the lines of which the figure is compoſed, lie all in the ſurface of the ſphere.

2. A ſpherical circle is ſaid to be inſcribed in a figure which lies wholly within the ſurface of the ſphere, or that figure is ſaid to be circumſcribed about



a ſpherical circle when the circle touches all the ſides of the figure.

THEOREM I.

In any ſpherical triangle A B C, compoſed of great circles, the difference of any two ſides A C and A B, is leſs than the third ſide B C.—(Pl. 4. fig. 7.)

Produce A B to V, and lay off D A upon C A = A B. (P. III. B. I.) The angle C D B = D B V (T. VIII. B. I.) muſt be greater than D B C. Hence (T. X. B. I.) C B is greater than C D.

THEOREM II.

If from the ends B, C, of the ſide B C of a great circle ſpherical triangle B A C, there be drawn two arches B D, D C, to a point D within the triangle, their ſum ſhall be leſs than the other two ſides B A and A C of the triangle.—(Pl. 5. fig. 22.)

Produce B D to E in A C. Then (by T. IX. B. I.) A B + A E are greater than B E. To each of theſe add E C; then B A + A C, are greater than B E + E C. (A. 8.) Again, C E + D E, are greater than D C, add D B; therefore, B E + E C, are greater than B D + D C; (Ibid.) much more then will B A + A C, be greater than B D + D C.

THEOREM III.

If from two given points A, B, on the ſphere, lying on the ſame ſide of a great circle P Q, two great circles A E, B E, be drawn to meet on and make equal angles A E Q, B E I, with the ſaid circle. The arches ſo drawn, taken together, ſhall be leſs than any other two, A G, B G, drawn from the ſame points, to meet on the ſame line P Q.—(Pl. 4. fig. 3.)

For let the arch B I V be perpendicular to P I Q, and let A E be produced to meet it in V. Alſo let V G be drawn.

Becauſe the angle V E I = A E G. (T. XIII. B. I.) V E I = B E I, (A. 1.) and ſince angle B I E = V I E, the arch V E = B E. (C. V.

T. XX. B. I.) Whence, alſo, $VG = BG$: (C. I. T. XX. B. I.) But $AV = AE + BE$, is leſs than $AG + VG$; (T. IX. B. I.) or its equal $AG + BG$.

COR. I. Let REL be any curve convex to PEG, and touching it in E: The ſum $BE + AE$, will ſtill be a minimum.

To any other point L, in REL, draw the arches BL, AL, including BG, AG: Then, $BL + AL$, are greater than $BG + AG$; (T. II. B. IV.) and, conſequently, much greater than $BE + AE$.

COR. II. Draw EH perpendicular to PQ: Then is angle $BEH = AEH$, when $BE + EA$ is a maximum.

THEOREM IV.

If in a given ſpherical triangle ABC, a point is to be determined, ſo that the ſum of all the three arches drawn from thence to the three angles, ſhall be the leaſt poſſible; the poſition of that point muſt be ſuch, that all the angles formed about it by thoſe arches, ſhall be equal among themſelves.—(Pl. 5. fig. 14.)

If it be denied, then let ſome point E, at which the angles B E A, C E A, are unequal, be the required one.

Upon the center A, through E, let the circumference of a circle D E be drawn, and let D be the point in it where the angles A D B and A D C are equal.

Becauſe B D + D C are leſs than B E + E C; (C. II. T. III. B. IV.) therefore, A D + B D + D C alſo are leſs than A E + B E + E C, which is abſurd; therefore, no point at which the angles are unequal can be the required one.

THEOREM V.

The greateſt great circle ſpherical triangle A I Q that can poſſibly be contained under two given arches, and any other arch joining their extremities, will be when the two given arches I A, A Q, make right angles with each other.—(Pl. 4. fig. 4.)

Lay off L A = A Q. (P III. B. I.) and draw through I and L, Q, the equal parallel circles I B, L Q.—(P. I. B. II.)

With A I as radius, and pole A, deſcribe the arch I E. Through any point E therein, and A, draw A E: Alſo through Q, E, draw Q E, and produce it to meet I B in B, and join B, O, and O, E.

(By C. I. T. V. B. II.) the triangle O I Q = O B Q is therefore greater than O E Q. To each of theſe join the triangle O Q A; then the trapezium O B Q A is greater than the trapezium O E Q A, and therefore much greater than A E Q; conſequently A I Q equal to the trapezium O B Q A, is alſo much greater than A E Q.

THEOREM VI.

Of all great circle ſpherical triangles A B C, A B D, having the ſame baſe, and the ſum of their other ſides the ſame; the iſoſceles one A C B is the greateſt. — (Pl. 5. fig. 7.)

Let the equal parallels F E D = A H B be drawn through D, and A, B; (P. I. B. II.) and let C H I be a great circle perpendicular to theſe, cutting A B in H, and E D in E, and join A, E; E, B.

The angles A E I, B E I, are equal; (C. III. T. XX. B. I.) and, therefore, A E + E B are leſs than A D + D B; (C. I. T. III. B. IV.) or than their equal A C + C B: And ſo the triangle A E B, or its equal A D B, (C. I. T. V. B. II.) is leſs than A C B; to each of theſe add the figure A H B I; then the triangle A C B I is greater than A D B I.

THEOREM VII.

If D V, E V, be equal circles on the ſphere given in magnitude and poſition, cutting another circle D E, whoſe center is G, given alſo in magnitude and poſition, in D d and E e reſpectively: Alſo, let A O B be a third circle equal to D V, E V, touching the circle D E in O, and interſecting D V and E V, ſo that A O may be equal to O B; and ſuppoſe the circles D V, E V, and A B, all either concave or convex to D E. The ſpherical triangle A V B will be a maximum when the point O falls on that part of the circumference of D E which is convex to the point V; and a minimum when the point O falls on that part of the circumference of D E which is concave to the point V.—(Pl. 4. fig. 8. 9.)

Draw any other arch C P Q I of equal curvature with A O B, (and alike concave or convex to G) to touch D E on the ſame ſide of its center G with A B, and interſecting V D in C, V E in i, and A B in P. Alſo, from B draw B F of equal radius, and convexity with A B touching D E in F, (P. IX. B. I.) and cutting C i produced in I.

Then A O = O B = B F, (C. IV. T. XXII. B. I.) and P O = O Q. Now, Q I = I F being leſs than B F or A O, we have A P = A O + O P, greater than P Q + Q I, and therefore much greater than P Q + Q i = P i.

Again, lay off m P = P i, L P = P B, and from m draw the arch m L of equal curvature, &c. with B P and B i. Then the triangle B P i = L m P (C. II. T. VI. B. II.) is leſs than the triangle A C P. To each of theſe add the trapezium C V B P, (Pl. 4. fig. 9.) and we have the triangle A V B greater than C V i. Alſo to each of the foregoing unequals add the trapezium A V i P, (Pl. 4. fig. 8.) and we have the triangle A V B leſs than the triangle C V i.

COR. I. Join D, E; then is the trapezium DABE a minimum when the arch AB falls between V and G, and alſo a minimum when G lies between the circle AB and V. This will appear evident by conſidering that DVE is a conſtant quantity.

COR. II. Draw RK any where between V and O, cutting AV in R, and VB in K: Then, will the trapezium ARKB be a maximum when the point O falls on that part of the circumference DOE, which is convex to the point V; and a minimum when the point O falls on that part of the circumference DOE, which is concave to the point V:

For RVK is a conſtant quantity.

COR. III. If the ſides DV, VE, (Pl. 4. fig. 15.) be tangents to the circle DE in D, E; then will AV+VB+AB be a conſtant quantity when the point O falls on that part of the circumference DE, which is convex to V: (T. V. B. III.) And, hence, it appears, that of all ſpherical triangles having equal perimeters, (limited as in the Theorem) and their vertical angles equal, the greateſt will be the iſoſceles triangle AVB.

COR. IV. Again, if the ſides DV, VE, (Pl. 4. fig. 13.) be tangents to the circle DE in D, E; then will AV+VB−AB be a conſtant quantity, when the point O falls on that part of the circumference DE, which is concave to V: (T. V. B. III.) Whence it follows that of all ſpherical triangles having the difference between the ſum of their ſides and baſe the ſame, and their vertical angles equal, the leaſt will be the iſoſceles triangle AVB.

THEOREM VIII.

Of two ſpherical triangles ABC, ABG, having two ſides AB, BC, of the one, equal to two ſides AB, BG, of the other, each to each reſpectively, the baſe of that ABC will be the greateſt which is ſubtended under the greateſt angle ABC.—(Pl. 2. fig 19.)

DEFINITION. By the angle which unequal circles on the ſphere form, I mean that which their tangents make with each other at the angular point.

Draw the arch GC, (P. I. B. I.) upon which let fall the perpendiculars AI and BH.

Since GH = HC (C. VI. T. XX. B. I.) GI = IH − GH is therefore leſs than IC =

I H + G H (H C), and therefore (C. IV. T. I. B. III.) A G is leſs than A C.

COR. I. If the ſide B C = B G, A B = A B, but the baſe A C greater than the baſe A G, then will the angle A B C be greater than A B G.

COR. II. From B let fall the perpendiculars B L, B V, on A C and A G reſpectively; then if the triangles A B C, A B G, be iſoſceles, and the baſe A C greater than the baſe A G, the perpendicular B L will be leſs than the perpendicular B V.—(Pl. 2. fig. 13.)

(By T. I. B. III.) The perpendicular B L is leſs than B O, (O being the point where B V meets A C), and therefore much leſs than B V.

COR. III. Hence, converſely, if B L be leſs than B V; (every thing elſe remaining as in Cor. II.) then will the baſe A C be greater than A G.

COR. IV. With radius B C = B G = B A and pole C, deſcribe the circumference A G C; then, if circles A C, A G, of equal curvature (all concave or all convex to the point B,) be deſcribed, to cut the circle A G C: thoſe whoſe perpendicular diſtance from B, as A C is leſs than thoſe as A G, will contain a greater intercepted arch A C than A G.

THEOREM IX.

Of all great circle ſpherical triangles A B C, D E F, having one angle B A C in the one, equal to one angle E D F in the other, and the ſides oppoſed to them equal, that will have the greateſt baſe A B, whereof the oppoſite angle A C B differs the leaſt from a right angle.—(Pl. 5. fig. 3. and 4.)

Let B G and E H be arches perpendicular to A C and D F, in which produced take H K = H E, G I = G B, and B M = E H; (P. IV. B. I.) alſo let C I and K F be drawn. (P. I. B. I.)

The angle I C G being equal B C G, (converſe C. VI. T. XX. B. I.) and the latter of theſe greater than E F H or K F H, by hypotheſis: Thence, is angle I C B greater than K F E; and, conſequently, B I is greater than K E: (T. VIII. B. IV.) Whence, $BG = \frac{BI}{2}$ is greater than $EH = \frac{KE}{2}$, or its equal B M.

Suppoſe, now, VP = MB, drawn perpendicular to AG, ſo as to meet AB in P; then the triangles AVP and DHE will be equiangular, (T. XVI. B. I.) and DE = AP, be leſs than AB.

THEOREM X.

If through the middle of the baſe P, of any ſpherical iſoſceles triangle AEB (of uniform curvature) an arch of equal radius, (and alike concave or convex) with the baſe be drawn, meeting the ſides AE, EB, in C and D reſpectively; it will be greater than the baſe AB.— (Pl. 4. fig. 5.)

Draw the circle AcBd touching AE and EB in A and B reſpectively, (P. II. B. III.) and cutting CD in c and d. Join P and the center of the circle O, and from O let fall the perpendicular OV upon CD.

The arch OV is leſs than PO; (T. I. B. III.) and, hence, cVd is greater than AB; (C. III. T. VIII. B. IV.) therefore, much more is CVD greater than AB.

COR. I. No arch of equal and ſimilar curvature with A B, can be drawn from C, between the points A, P, on the ſame ſide of O, and intercepted by A E, E B, to be equal or leſs than A B.

THEOREM XI.

Of all arches of equal curvature (alike concave or convex to E), which can be drawn to cut off equal areas from a given iſoſceles ſpherical triangle E L R, (of the ſame curvature with theſe arches) that A B is the leaſt which makes the triangle A E B iſoſceles.—(Pl. 5. fig. 6.)

Suppoſe C O D drawn, meeting A B in O, and cutting off the area C E D = A E B; and which, to be equal or leſs than A B, muſt paſs on that ſide of the middle point P thereof, which is next to B, when the point D falls below B. (C. I. T. X. B. IV.) From O upon O D lay off O V = O A, and join A, V, with a circle of equal curvature with A B. From G, the middle of A V, draw the arch G O paſſing through O, and meeting A E in F. Through V, F, draw a circle of equal curvature with A E, (P. VI. B. I.) forming the iſoſceles triangle A F V.

Again, through E P draw the great circle E P e, and produce O G to Z:

Becauſe E P paſſes through the center of the circle A B, (C II. P. V. B I.) and O G through the center of the circle A V, (C II. P. V. B. I.) which cuts the former in O, their diameters will meet ſomewhere in the arch E P e as at Z: Whence it is plain that Z O prolonged will cut A E at F above the point E.

Let Q, I, be the points where V F meets E D and A B reſpectively:

Becauſe angle O A G = G V O, (T. VIII. B. I.) and angle C A V = I V A, and the ſide A V common to the two triangles C A V, I V A; the ſide A I will be equal V C; (converſe C. III. T. VII. B. I.) and C O = V C − V O = A I − A O = O I, and alſo taking away the triangle A O V, the area of the triangle A O C = V O I. (A. 5.) Hence, if from the equal triangles A E B, C E D, we take the trapezium E C O B, there remains the triangle B O D = A O C = V O I; (A 1 and 5.) and taking away O B Q V the triangle B Q I = V Q D.

Again, becauſe the area B Q I = V Q D, the point I muſt lie between F and Q in the curve QF, and ſince the arch QF lies wholly beyond BQ, the point I will alſo fall in AB produced beyond B, and conſequently the point V falls in OD between O and D; hence AB=AO+OB being leſs than AO+OI= CO+OV will much more be leſs than CO+OV+VD=CD.

COR. I. Through C and P draw the arch CPK of equal and ſimilar curvature with COD; then is the triangle CEK greater than the triangle AEB.

Since PO contains a ſpace, the point K muſt fall below D; and, conſequently, the triangle CEK is greater than CED, and, therefore, greater than its equal AEB.

THEOREM XII.

Of all ſpherical triangles AEB, CED, of uniform curvature, ſtanding upon equal baſes AOB, COD, both concave or both convex to the vertical angle E, and which is alſo equal in each; the iſoſceles triangle AEB is the greateſt. —(Pl. 4. fig. 10.)

Suppoſe the point C to fall between A and E: And from C let fall the perpendicular CP upon EB. Through C draw C o I, cutting off the triangle E C I = A E B.

The arch C I is greater than A B, or its equal C D; (T. XI. B. IV.) and, therefore, the point D falls between I and P; (C. IV. T. I. B. III.) and, conſequently, A E B = E C D + I C D is greater than E C D.

THEOREM XIII.

Of all arches E F, G H, of equal curvature, all concave or all convex to a given point B, which can be drawn through a given point D within the iſoſceles triangle R B L, (of the ſame curvature with theſe arches) that E F which is biſected by the given point D, forms the leaſt triangle E B F with the arches B R, B L. — (Pl. 5. fig. 5.)

From B let fall the perpendicular B P upon R L, and let the point D be on that ſide of B P next to B L. Draw D A perpendicular to E F, meeting E B in A, and compleat the iſoſceles triangle E A F. (P. VI. B. I.) —

By what has been proved, (T. XI. B. IV.) A D produced will meet B P in ſame point Z; and conſequently the point A falls in E B beyond B. Alſo, if the point H lie above the point F; then G H produced will meet A F in I, beyond B F next to A. Or, if the point H lie below the point F, then G H will be cut by A F within F L, next to Z.

Firſt, let the point I fall in A F beyond B F, next to A.

The triangle A E F is leſs than G A I, (C. I. T. XI. B. I.) From each of theſe take the triangle B A F; then E B F is leſs than G B H – H F I, and therefore much leſs than G B H.

Secondly, let the point I fall in F I within F L, next to Z.

The triangle E A F is leſs than G A I, (C. I. T. XI. B. IV.) From each of theſe take the triangle B A F, then the triangle E B F is leſs than the trapezium G B F I, and therefore much leſs than the triangle G B H.

THEOREM XIV.

Of all figures on the ſphere contained under the ſame number of great circles, and inſcribed in the ſame circle; that is the greateſt whoſe ſides are all equal. —(Pl. 4. fig. 6.)

For, if poſſible, let ſome polygon A B C F E, whoſe ſides C F, F E, are unequal, be the greateſt.

Let C D E be an iſoſceles triangle deſcribed in the ſame ſegment with C F E, and thro' C E and F deſcribe the equal parallels C E, F I. (P. I. B. II.)

Becauſe the point D lies above F, it will alſo lie above any other point in the parallel I F. Let O, therefore, be the point where I F cuts C D and join O E.

The triangle C D E being greater than C O E, is alſo greater than its equal C F E. (C. I. T. V. B. II.) Whence, the whole polygon A B C D E will be greater than A B C F E, which is abſurd. Therefore, the polygon is the greateſt when the ſides are all equal.

THEOREM XV.

Of all figures on the ſphere contained under an equal perimeter of great circles, and the ſame number of ſides, the greateſt is that whoſe ſides are all equal.—(Pl. 5. fig. 8.)

For, if A B C D E be the greateſt poſſible, the triangle C D E muſt manifeſtly be greater than any other triangle C F E upon the ſame baſe, and having the ſum of its other ſides the ſame. But (T. VI. B. IV.) the greateſt triangle when the baſe and ſum of the ſides are given, is that whoſe ſides are equal; conſequently D C = D E. In the ſame manner it will appear that B C = C D, &c.

THEOREM XVI.

If all the ſides of a ſpherical polygon, except one be given in length, curvature, and convexity, and their poſition be required, ſo as to make the polygon itſelf the greateſt poſſible; their poſition muſt be ſuch, that two arches drawn from the extremes of the unknown ſide to any angle of the polygon, ſhall form a right angle.—(Pl. 5. fig. 9. 10.)

For if you would have the polygon ABCDEF to be the greateſt poſſible, and yet ADF, ſubtended by the unknown ſide AF, not a right angle; then let PSO be a right angle contained under PS=AD, and OS=FD; and upon PS and OS, let the figures PSRQ and OST be deſcribed equilateral and equal to ADCB and FDE.

The triangle PSO is greater than ADF, (T. V. B. IV.) therefore PSRQ being equal to ADCB, and OST equal to FDE, the whole polygon PQRSTO is alſo greater than the whole polygon ABCDEF, which is abſurd.

The properties of that curve deſcribed in plano, whoſe area is a maximum, when its perimeter is given, have long been determined; but thoſe which will determine the like curve, when deſcribed on the ſurface of the ſphere, (I preſume) have hitherto remained uninveſtigated. I ſhall therefore proceed to diſcover its equation. In order to which it will be requiſite, to ſolve the following problem.

PROBLEM I.

Required the Locus BCVA, on the

ſphere of the great circles, VA, AB, and CA, CB, drawn from two given points A, B, to meet two and two ſuch, that the angles AVB, ACB, thereby formed, ſhall be always equal. —(Pl. 4. fig. 14.)

Suppoſe BA, the principal axis of the curve, paſſing through the given points A and B: Alſo, ſuppoſe VQ, its ſemiconjugate, perpendicular thereto, and biſecting it in Q: And from any other point C, in the curve, let fall the perpendicular CP upon BA. Put tangent angle C $=t$, tangent AB $=$ T, tangent BP $=x$, and tangent PC $=y$; then we have ſine of PC $=\frac{y}{\sqrt{1+y^2}}$. Whence, per Trigonometry, PC, $\frac{y}{(\sqrt{1+y^2})}$: radius (1) :: tangent PB (x) : tangent angle PCB $=x\frac{\sqrt{1+y^2}}{y}$. Now, by Corollary to Propoſition III, page 56, Simpſon's Trigonometry, tangent angle ACP $=\frac{t-\frac{x}{y}\sqrt{1+y^2}}{1+\frac{tx}{y}\sqrt{1+y^2}}$, and tangent AP $=\frac{\mathrm{T}-x}{1+\mathrm{T}x}$, where radius is ſuppoſed unity: Hence, again, per Trigonometry $\frac{T-x}{1+Tx} : x :: \frac{t - \frac{x}{y}\sqrt{1+y^2}}{1+\frac{tx}{y}\sqrt{1+y^2}} : \frac{x}{y}\sqrt{1+y^2}$; that is, $\frac{T-x}{1+Tx}.\frac{x}{y}\sqrt{1+y^2} = x.\frac{t-\frac{x}{y}\sqrt{1+y^2}}{1+\frac{tx}{y}\sqrt{1+y^2}}$, and multiplying by $1+\frac{tx}{y}\sqrt{1+y^2}$, and dividing by x we obtain $\frac{T-x}{1+Tx}\sqrt{\frac{1}{y^2}+1} + \frac{Tt-tx.}{1+Tx}.\overline{\frac{x}{y^2}} + x = t - x\sqrt{\frac{1}{y^2}+1}$ or by tranſpoſition $\overline{\frac{T-x}{1+Tx}+x}.\sqrt{\frac{1}{y^2}+1} = t - \frac{Tt-tx.}{1+Tx}\overline{\frac{x}{y^2}+x}$; laſtly, multiplying by $1+Tx$ and proper reduction we have $\overline{T+Tx}.\sqrt{\frac{1}{y^2}+1} = t - \frac{Ttx}{y^2} + \frac{tx^2}{y^2} + txx$, the required equation.

COR. I. Suppoſe $x=0$, then the equation becomes $T.\sqrt{\frac{1}{y^2}+1} = t$, or $y = \pm T\sqrt{\frac{1}{t^2-T^2}}$, which ſhews that the curve is convex to the axis A B at the point B.

COR. II. When the angle A C B is a right angle, the value of t is infinite. Hence, rejecting all thoſe quantities in which it is not found, we have $o = t - \frac{Ttx}{y^2} + \frac{tx^2}{y^2} + tx^2$ or dividing by t, &c. $y^2 + x^2y^2 = Tx - x^2$; and, laſtly, dividing by $1 + x^2$ we have $y^2 = \frac{Tx - x^2}{1 + x^2}$.

THEOREM XVII.

Of all ſpherical figures I A V C B, E F G H L, contained under equal perimeters, that is the greateſt whoſe equation is $y^2 = \frac{Tx - x^2}{1 + x^2}$, where T repreſents the tangent of A B, an arch of a great circle, y the tangent of C P, an arch drawn perpendicular thereto, and x the tangent of P B, the arch cut off, A B by C P.—(Pl. 5. fig. 11. 12.)

For ſince A B is ſuppoſed its principal diameter, if we ſuppoſe B G taken equal thereto, and the perimeter E F G taken equal to the perimeter AVCB, then the area AVCB will be greater than EFG by (T. XVI. B. IV.) and COR. II. of the preceding problem; and in like manner ABI will be greater than EGHL; therefore IAVCB muſt neceſſarily be greater than EFGHL.

THEOREM XVIII.

Suppoſe BDKIEC any conic ſection concave to its principal axis IP, and ſuppoſe LOK any other axis, and DOE its conjugate biſected in O, and draw BLC parallel to DOE. Alſo, ſuppoſe the curve by its revolution round its principal axis IP, to deſcribe the ſolid BDKECg; produce OK, which (by conics) will meet tangents to the curve drawn from D and E in the ſame point V; and through DE and BC draw planes perpendicular to the plane DVE, and conſequently forming ſimilar curves DeE, BgC.

Compleat the cone TDVEGQ, and draw Da and Er parallel to KL; alſo compleat the cylinder DeEra; then, if we ſuppoſe $DV = 2DT$, and conſequently $VE = 2EG$, I ſay the cylinder Dr will be the greateſt that can poſſibly be inſcribed in the ſolid BDKECg. —(Pl. 5. fig. 20.)

Alſo, let BDKEC be any curve concave to its principal axis OK; then, ſuppoſing IP perpendicular to the ſections DeE, BQG, in which caſe PK and PI coincide; then, ſuppoſing every thing elſe as before, the cylinder Dr will, in this caſe, alſo be a maximum. —(Pl. 5. fig. 18.)

It may eaſily be demonſtrated (ſee Simpſon's Geo. Theo. 19. maxima and minima) that Dr is the greateſt cylinder that can be cut out of the cone TDVEGQ, and is therefore much more the greateſt that can be cut out of its inſcribed ſolid BKECg.

THE

CONSTRUCTION

OF

SPHERICAL PROBLEMS.

BOOK I.

PROBLEM I.

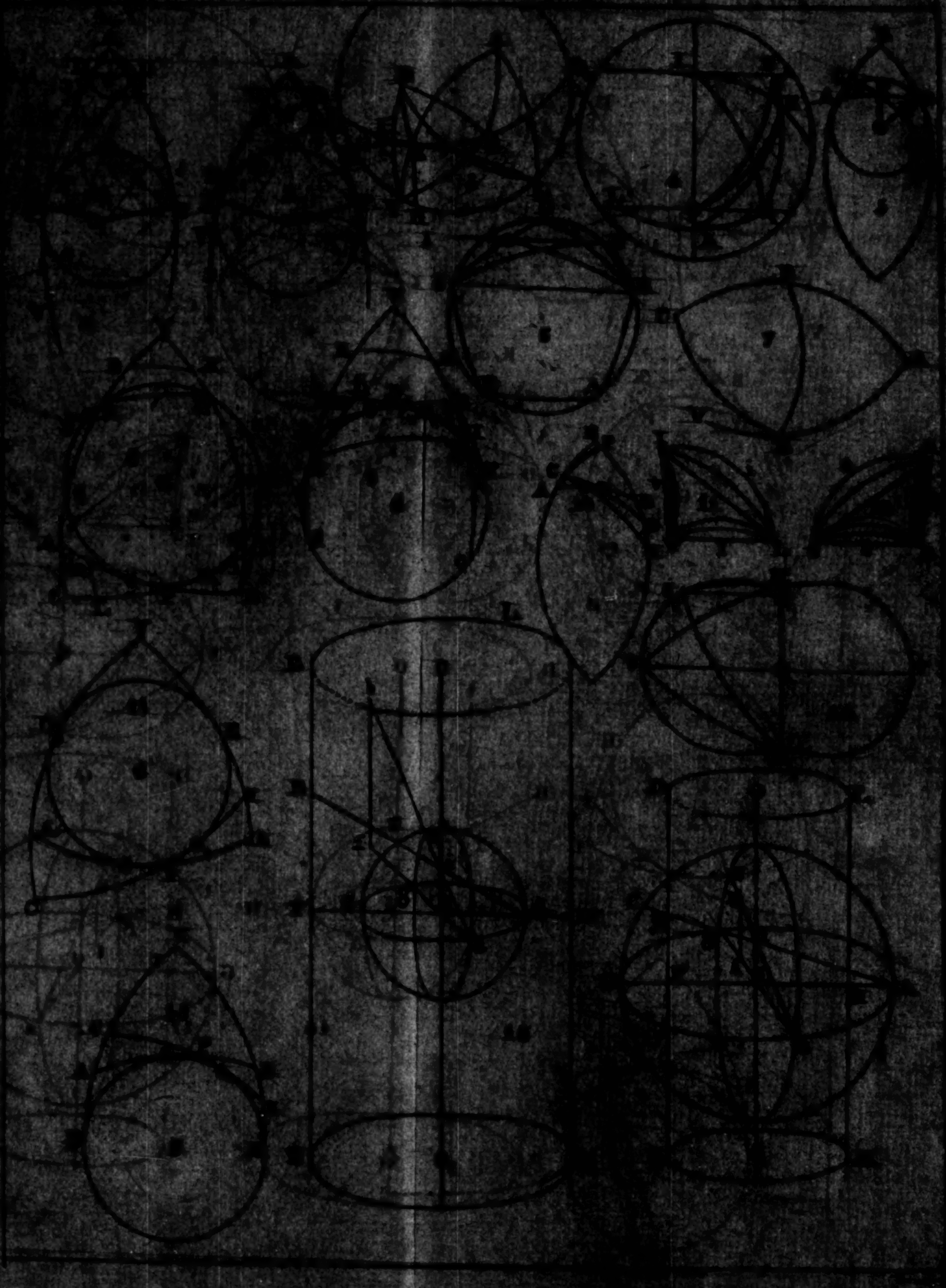
TO reduce a great circle ſpherical polygon ABCDE, to another GCDE of equal area and fewer ſides.—(Pl. 6. fig. 4.)

CON. To take away the angle B, produce the next ſide EA towards A.

Through AC and B reſpectively, draw the equal parallel circles AC, BG. (P. I. B. II.) Where CB meets AG in G, join GC, (P. I. B. I.) which takes in the triangle CAG inſtead of its equal CAB; (C. I. T. V. B. II.) and the polygon becomes GCDE.

COR. I. By thus taking away one angle after another; any great circle polygon may at laſt be reduced to a triangle.

Plate [illegible]



COR. II. If the ſum of the angles in two ſpherical figures compoſed of great circles are equal, their areas will be equal and the converſe.

Draw the arch AC. (P. I. B. I.)

Since the triangles AGC, ABC, are equal, (C. II. T. V. B. II.) the ſum of their angles are equal. (T. IX. B. II.) To each of theſe equals add the angles of the figure ACDE, then will the ſum of the angles in the figures EAGCD, EABCD, be equal. (A. 4.)

The converſe is evident.

PROBLEM II.

A ſpherical great circle triangle ABC, being given, to reduce it to another AED of a different baſe and equal-area.—(Pl. 6. fig 12.)

CON. Let AE be the required baſe, being in the ſame arch with AB. Draw the equal parallel circles CE, BD; (P. I. B. II.) where BD cuts AD in D; join D, E, with the great circle DE; then is the triangle ADE equal to the triangle ABC.

DEM. The triangle DBE is equal DBC;

(C. I. T. V. B. II.) to each of theſe add the triangle ADB; then is the triangle ADE = ABC.

PROBLEM III.

With two given arches A, B, to make a ſpherical rectangle.—(Pl. 6. fig. 2.)

ANALYSIS. Suppoſe the thing done, and that IDCE is the rectangle required, having the ſides IE = DC = A, and ID = EC = B, produce DC and IE until they meet in V and L.

Since DC is given, VD = LC = LE = IV are all given; and becauſe ID is given, all the ſides of the triangle IVD are given; and hence the following:

CON. With the given lines IV, VD, and ID, make the triangle IVD; (P. I. B. III.) and on VI, VD, produced, lay off IE and DC reſpectively, each equal the arch A. Laſtly, through E, C, draw the great circle EC, and the thing is done.

The demonſtration is evident from the analyſis.

PROBLEM IV.

To draw a circle VIOD on the

ſphere of given curvature, to cut two other circles L I, O B, given in magnitude and poſition, in given angles V I L and D O B.—(Pl. 5. fig. 13.)

ANALYSIS. Suppoſe the thing done, and through A, the centre of V I O D; and C, D, the centers of the circles L I and O B, draw the arches A C V, A Q D, meeting the circumference V I O D in V and D reſpectively; and join C, Q.

Since the angle L I V is given, the perpendicular C V upon i V I is alſo given, (SCHOLIUM P. X. B. I.) as alſo the perpendicular Q D upon o D O.

Again, ſince A V is given, A C = A V – V C, and A Q = A D (A V) – Q D, are each given; and the arch Q C is alſo given, by hypotheſis, whence the whole triangle A C Q is known. Hence the following

CON. Upon C Q, conſtruct the given triangle A C Q, (P. I. B. III.) and produce the ſide A C till A V is equal to the given radius of V I. Laſtly, with radius A V, and center A, deſcribe the circumference V I O D, which will cut L I in I, and D O in O, the points required.

COR. I. Hence to conſtruct a ſpherical triangle P I O, of which all the angles and curvature of the ſides are given. Join one of the given angles I P Q, and by the foregoing propoſition draw the arch V I O of given curvature, meeting the former arches in given angles at I and O and the thing is done.

PROBLEM V.

To make a great circle ſpherical ſquare A B D C, equal to a given great circle ſpherical polygon E F G H I.—(Pl. 7. fig. 3. 4.)

CON. On the arch A D, deſcribe the perpendicular O B, (P. VII. B. I.) and draw B D, cutting O B, O D, each in an angle O B D = O D B = one-eighth of the ſum of the interior angles of the polygon E F G H I. (C. I. P. IV.)

Produce B O and O D, and make the triangles C O D, C O A, A O B, each equal B O D, (P. I. B. III.) and the thing is done.

DEM. The angles O B D, O D B, O C D, O A C, &c. are all equal by conſtruction, and, therefore, there ſum, viz. eight times

OBD, equal the ſum of the interior angles of the polygon EFGHI: Whence, the areas of the two figures are equal. (C. II. P. I.)

Alſo, ſince the angles at O, are all right angles, and the ſide BD = DC = AC = AB; the figure ABDC is a ſquare.

PROBLEM VI.

With two given angles A, B, to make a ſpherical parallelogram, having one of its ſides equal to a given arch CD, of a given circle.—(Pl. 6. fig. 8.)

ANALYSIS. Suppoſe the thing done, and that EFCD is the required parallelogram, having the given ſide CD, and the angle FCD = A, and angle CDE = B.

Produce FC and DE, both ways to meet in V and I: Then, becauſe, angles FCD, EDC are given; DCI is alſo given as well as CDI; and, hence, the following

CON. Upon CD, lay off the angles DCI, CDI, and produce CI and DI, till they meet in I and V. From V, lay off VF = DI, and VE = CI. Laſtly, join F, E, with a circle of equal curvature with D C: (P. VI. B. I.) Then is E F C D, the parallelogram required.

PROBLEM VII.

Upon a given arch F G, to make a ſpherical parallelogram equal to a given great circle ſpherical polygon B A C D E, and having an angle equal to a given angle V.—(Pl. 5. fig. 15. 17. 23.)

CON. On F G, from F, lay off the given angle G F K = V. From the ſum of the angles A, C, D, E, B, of the polygon, lay off twice the angle V. Biſect the remainder, and call its half H. From G, lay off F G L = H, and compleat the parallelogram L K F G, (P. VI.) and it is done.

DEM. By conſtruction, angle F = V, the given one; alſo, F + G + L + K = A + C + D + E + B; therefore, the areas K F G L, A C D E B, are equal. (C. II. P. I.)

PROBLEM VIII.

To draw a circle I V of given curvature on the ſphere, to touch two circles

I A, B V, given in magnitude and poſition.—(Pl. 5. fig. 19.)

ANALYSIS. Suppoſe that O, Q, are the centers of the given circles I A, B V, reſpectively. From O to Q draw the great circle O Q; (P. I. B. I.) and join O, I; V, Q, with great circles, and produce the arches O I, V Q, till they meet in P, which (becauſe theſe arches are both perpendicular to I V) will be the pole of I V. (C. II. P. V. B. I.)

Again, ſince O I and V Q, as well as I P = P V, are given, P Q = P V ± V Q, and P O = P I ± I O, are alſo given; and alſo the triangle O P Q is given. Hence the following

CON. Having formed the triangle O P Q (P. I. B. III.) about P, as a pole with the radius P I, deſcribe the circle I V, which will touch the circles I A, B V, in I and V, the points required as is plain from the analyſis.

PROBLEM IX.

In a given leſs circle A D B C, to apply an arch A B, equal in length and curvature, to a given arch R S, (whoſe chord is leſs than the chord of the diameter of A D B C) and which ſhall paſs through a given point P.—(Pl. 5. fig. 16.)

CON. Inſcribe the arch C D = R S, in A D B C, upon which from the center O of A D B C let fall the perpendicular O F. With radius O F and center O deſcribe the circle F E. From P draw P E touching F E in E, (P. IX. B. I.) and produce it both ways to meet the circumference in A B, ſo ſhall A B be the arch required.

DEM. Becauſe B E and C F are both tangents to E F at E and F; F D = C F = B E = A E, (C. I. P. IX. B. I.) conſequently A B = C D = R S.

PROBLEM X.

Having two concentric circles B A C and L R Q, of the ſphere given in magnitude and poſition; and alſo, a given point P; it is required to draw an arch C Q D, of given curvature, through P, ſuch that the part A Q, that is intercepted between B A C and L R Q, may be of a given length.——(Pl. 6. fig. 3.)

ANALYSIS. Suppoſe the thing done, and from the common center O, draw Q O, A O, O B; (P. I. B. I.) alſo let fall upon A Q, the perpendicular O E, which will biſect the arch A B. (C. II. T. XX. B. I.) Now, becauſe the arches A O, O Q, and A Q, are given, the triangle A Q O, as well as the perpendicular O E, are all given; and, hence, A E and its double A B is given; whence the propoſition is reduced to the laſt.

PROBLEM XI.

Through a given point P, on the ſphere, to draw an arch A B, of given curvature ſuch, that the parts thereof A P, P B, intercepted by that point, and the circumference of a given circle A B C, ſhall have a given difference D E.—(Pl. 6. fig. 1.)

ANALYSIS. Through O, Q, the reſpective centers of the circles A C B and A P B, draw the great circle Q O I, meeting A P B in I: Alſo, join P, O, and P, Q:

Then, ſince Q I is perpendicular to A B, A I is equal to I B; (C. VII. T. XX. B. I.) and, conſequently, P I equal to half the given difference between P B and A P. Again, ſince P Q is given, and the angle I given, the triangle P Q I is given; and, hence, the following

CON. On p i, equal in length and curvature to P I, erect the perpendicular i q; and from p, with radius p q equal to P Q, deſcribe an arch meeting i q in q: Alſo, with center p and radius p o, equal to the given arch P O, deſcribe an arch cutting i q in o.

Again, upon the baſe P O make the triangle P O Q = p o q. (P. I. B. III.) Laſtly, with the radius P Q and center Q, deſcribe the arch A P I B cutting A C B in A B, and the thing is done.

PROBLEM XII.

B E is a given leſs circle, C D an arch drawn from the center C, given in magnitude and poſition; it is required to draw an arch E F from the circumference B E, and meeting the circle M N, given in magnitude and poſition, ſo that joining C, E, and D, F, the figure CEDF ſhall be a parallelogram—(Pl. 7. fig. 6.)

CON. From D, with radius DF=CB, deſcribe an arch, meeting MN in F. Alſo with radius FE=CD, and center F, deſcribe an arch, cutting BE in E. Laſtly, join EF, and C, E, and the thing is done.

DEM. By conſtruction, DF=EC, and EF =CD, conſequently the figure CEDF is a parallelogram. (D. 3. B. II.)

PROBLEM XIII.

Suppoſe FC any curve formed on the ſphere, ED any circle given in magnitude and poſition, and KER a great circle paſſing through K, the center of ED, and any given point R: It is required to draw a great circle KDC, paſſing through K, ſuch that the arch DC, interſected between the curve FC and the arch ED, may be equal to a given arch AB.—(Pl. 7. fig. 5.)

CON. From K lay off KP upon KR, equal half a great circle; and from center P, with radius PC=KP−AB−EK, deſcribe an arch, cutting the curve FC in C; through K, C, P, draw the arch PCDK, and the thing is done.

DEM. PC+CD+DK=KP=PC+AB+EK by conſtruction; and ſince DK=EK, by the nature of the circle, therefore CD=AB.

PROBLEM XIV.

Through a given point P, between a curve CF and a great circle AB, both given in poſition, to draw a great circle FPE, ſo that the parts thereof, FP, PE, intercepted by that point and the two curves, ſhall be equal. — (Pl. 6. fig. 6.)

CON. Through P to any point G in AB, draw PG, on which, produced, take PH=PG; and through H draw the great circle bH, making the angle PHb=PGA, and meeting the curve in F. Through F draw the great circle FPE, meeting AB in G, and the thing is done.

DEM. The triangles FHP, GPE, having the ſide PH=GH, and the angles P=P, and FHP=PGE, are ſimilar and equal, (T. IV. B. I.) conſequently FP=PE.

PROBLEM XV.

In a given circle of the ſphere CIHK to deſcribe three equal circles E, F, and G, which ſhall touch one another; and alſo, the periphery of the given circle. —(Pl. 5. fig. 21.)

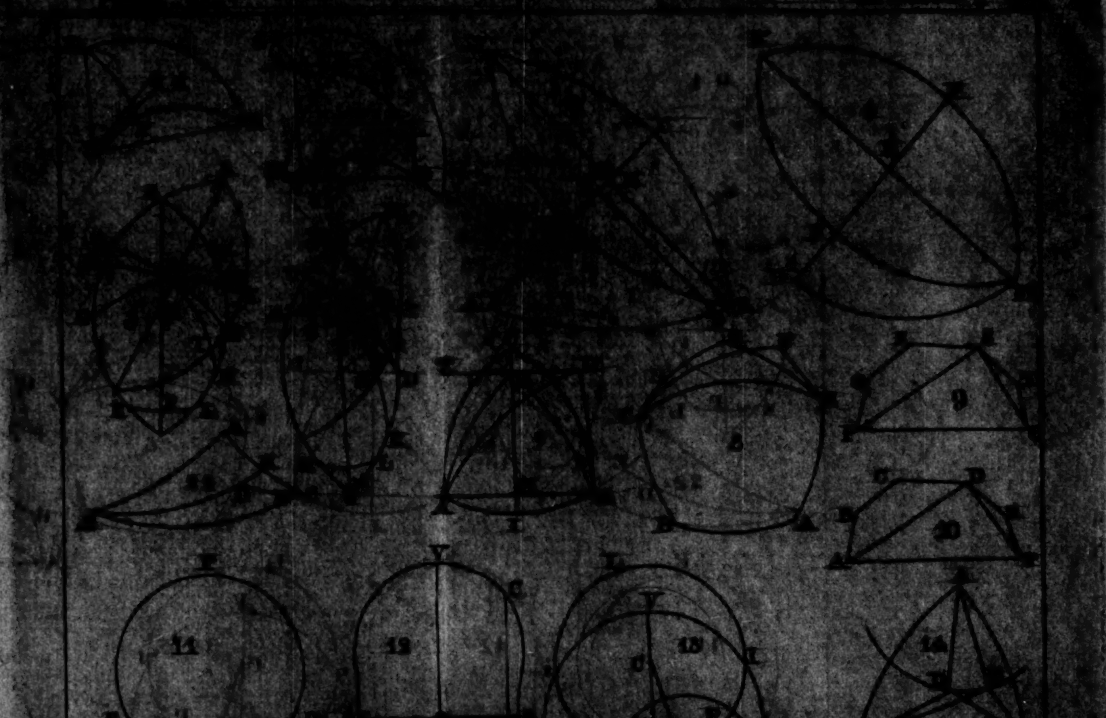
ANALYSIS. Suppoſe the thing done, and from C, the center of HKI; through the centers F, E, G, of the inſcribed circles, draw the arches CFI, CEH, and CGK: Alſo, from C, through m, n, where the circles FE and EG reſpectively touch each other, draw the arches CmV and CnO, meeting a ſpherical tangent at H, in V and O reſpectively: Then, becauſe the angles ICH, HCK, are biſected by CV and CO reſpectively, the angle CVO will be equal ICH =HCK=KCI=60°; and ſince the perpendicular HC is given, the triangle VCH, as well as VCO, is given. And by a ſimilar proceſs, may any poſſible number of equal circles be inſcribed in any given circle.

PROBLEM XVI.

To find the center of a circle L, to paſs through two given points A, B, on the ſphere, and cut off from a circle FDP, given in magnitude and poſition, a given arch FD. — (Pl. 6. fig. 5.)

ANALYSIS. Suppoſe the thing done, and draw FED, the chord of the required arch FD, and in any part of the circle FDP, lay off the chord GHI = FED; and from the center Q, of the circle FDP which is in the plane FDP, deſcribe a circle HE to touch CHI in H; and, conſequently, its equal FED in E: Alſo, through A, B, draw the chord AB, which produced, will meet ED alſo produced in ſome point C. For the points A, F, D, B, being all in the circle FDB, are conſequently in the ſame plane ADC. Again, from C to any point P, in the circumference FDP, draw the right line CPO which, becauſe the points F, D, C, P, are in the ſame plane, will cut the circumference FDP ſomewhere in O.

Plate



Again, ſince O C. P C = F C. D C = A C. B C, (C. 36. III. E.) the points O, A, B, P, lie all in the circumference of the ſame circle; hence, the following

CON. Through A, B, draw any circle cutting F D P in O and P, and through O, P, draw the chord O P meeting A B produced in C. From C, draw the right line C D E F, a tangent to the circle H E, deſcribed as before directed; then a circle drawn through A, B, D, F, is that required.

PROBLEM XVII.

To find the center of a circle L, to paſs through two given points A and B on the ſphere, and to touch a circle O E P given in magnitude and poſition. —(Pl. 6. fig. 7.)

CON. Through A, B, draw any other circle O A B P, cutting the circle O E P in O and P: Alſo, through A, B, draw the chord A, B, C, and through O, P, the chord O P, which (becauſe the points O, A, B, P, are all in the ſame plane) will meet the line A B C in ſome point C. From C, draw C E a tangent to PE in E. Laſtly, through the points A, E, B, (5. IV. E.) draw the circle AEB, and the thing is done.

DEM. By (C. 36. III. E.) AC.BC= OC.PC=EC²: Therefore, EC is a tangent to both ABE and OPE at E; (36. III. E.) and, conſequently, theſe circles touch each other at that point.

PROBLEM XVIII.

Through a given point P, on the ſphere, to draw a circle PVI, to touch two equal circles EVF, EIF, given in magnitude and poſition.—(Pl. 6. fig. 11.)

ANALYSIS. Suppoſe the thing done, and through E, F, the given interſection of the two circles, draw the chord FE, and produce it till it meet the equal tangents VA, AI, drawn from V and I reſpectively. From P, let fall the perpendicular PB upon EF, and produce it till Bp=BP: Then, ſince the points P, p, are at equal diſtances from FBE, and both given; a circle paſſing through P, p, and touching FIE in I, will alſo touch FVE, whence the problem is reduced to the laſt.

PROBLEM XIX.

Through a given point P, to draw a circle P V I on the ſphere, having its center L in a great circle F L E, given in poſition, and touching another circle F V E in I, given in magnitude and poſition. — (Pl. 6. fig. 11.)

ANALYSIS. Suppoſe the chord F E drawn as before, and from Q, the center of F V E, let fall the perpendicular Q C, and produce it till Q C = C O. Laſtly, with radius O I = Q V, deſcribe the circle F I E, which will conſequently meet F V E in F and E, whence the problem will be reduced to the laſt.

PROBLEM XX.

From two given points A, O, on the ſphere, to draw two arches A L, O L, to meet in a great circle I L, given in poſition, ſuch, that their ſum A L + O L, or difference O L − A L, ſhall be a given quantity. — (Pl. 6. fig. 9. 14.)

CON. About O, as a pole with the given ſum or difference E O, deſcribe a circle E B, and (by the laſt problem) draw the circle A E paſſing through A, to touch E B, and have its center in L I. Laſtly, join A, L, and L, O, and the thing is done.

DEM. Becauſe A L = L E; then, L O ± L A = L O ± L E = E O, by conſtruction.

LEMMA.

If B D, b d, be any two lines in the ſame plane B D b d, touching the convex ſuperficies of a given cone B A I on the ſame ſide of the axis, or the produced cone B A I, b A Q, on oppoſite ſides of the axis, in any two points B, b; then will the line B b lie wholly within the ſurface of the cone, and (produced if neceſſary) paſs through its vertex A.—(Pl. 6. fig. 13.)

DEM. Becauſe the plane B D d b touches the ſuperficies of the cone B A I on the ſame ſide of the axis O Q, and on oppoſite ſides of it in two points B, b, it muſt neceſſarily touch it in all the points of the ſtraight line B b, lying between B and b; and conſequently the line B b muſt lie wholly in the convex ſurface of the cone; and ſince all ſtraight lines lying in ſuch a ſurface muſt meet in the vertex, the line Bb will paſs through the point A.

PROBLEM XXI.

Through a given point P on the ſphere, to draw a circle PBb, to touch two circles BI, bQ, given in magnitude and poſition.—(Pl. 6. fig. 16. and Pl. 7. fig. 1.)

ANALYSIS. Suppoſe the thing done, and through O, V, the centers of the circles BI, bQ, deſcribe a great circle, cutting their circumferences in Ii and qQ reſpectively. Alſo through I, Q, and i, q, draw the ſtraight lines IQ, iq, meeting in A the vertex of the cones BAI, bAQ. (T. XII. B. III.) Join P, A, with a ſtraight line, meeting the ſphere (produced if neceſſary) in p. Alſo draw Bb, which, becauſe the lines BP, bp, lie in the ſame plane, and touch the conic ſuperficies BAI in B, b, will (produced if neceſſary) paſs through A by the preceding LEMMA.

Here then it is plain that the points BPAb are all in the ſame plane; and the point A being given, (T. XII. B. III.) if through P, A, we draw P A, meeting the ſphere, (produced if neceſſary) in p; then ſince I A. Q A = B A. b A = A P. A p, (C. 36. III. E. and 35. III. E.) that point p will alſo be given; which, lying alſo in the plane P B b A, muſt conſequently fall in the circumference of the required circle P B b.

Again, if on the middle D of a great circle paſſing through the given points P, p, we erect a perpendicular D L, this will meet arches drawn through B O and b V in the center L of the circle B b P, (Converſe C. VII. T. XX. B. I.) and D L will be given in poſition.

Laſtly, ſince B O + O L = L V + V b, (Pl. 6. fig. 16.) then L V − O L = B O − V b, a given quantity; and ſince L V = L O + O B + V b, (Pl. 7. fig. 1.) conſequently L V − L O = O B + V b, is alſo given. Whence this problem is evidently reduced to the laſt.

PROBLEM XXII.

To draw a circle E on the ſphere, to touch three given circles A, B, D.—(Pl. 7. fig. 2. 10. and Pl. 6. fig. 10.)

CON. With radius B G = A H ± B L, the

ſum or difference of the radii of the greateſt and ſecond circles A and B, and center B, of the ſecond, deſcribe the circle G g. Alſo with radius D F = A H ± D I, the ſum or difference of the radii of the greateſt and leaſt circles A and D, deſcribe the circle F f.

Again, through A, the center of the greateſt circle, deſcribe the circumference A G F, touching the circles G g, F f, in G and F, (by the laſt problem.) Laſtly, with its center E, and radius H E = A E ± H A, deſcribe the circle H I L, and the thing is done.

DEM. Join A, E; E, F, and E, G; and draw F D, A H, and B G; becauſe A F touches F f at F; the points E, F, D, are in the ſame great circle, (C. III. T. XVII. B. I.) and ſince, by conſtruction, the part of F D contained between the circles D and F f is equal, H A = F I, alſo, by conſtruction; the point I will fall in the circumference of D, in the ſame great circle F D; and, conſequently, H I will touch the former at I. And in like manner it will be proved, that H I will touch the circle B at L. Laſtly, ſince H E paſſes through A, and is equal A E ± H A; H I alſo will touch A H at H.

THE
CONSTRUCTION
OF
SPHERICAL PROBLEMS.

BOOK II.

PROBLEM I.

GIVEN one ſide A C, an angle adjacent A C B, and the ſum of the remaining angles A and B, to determine the great circle ſpherical triangle A B C. — (Pl. 7. fig. 7.)

ANALYSIS. Suppoſe the thing done, and compleat the parallelogram A D B C, and produce D A, B C, to meet in I and V; then, becauſe angle D = C and B = A, alſo their ſum S = D + B + C + A being given, we have angle $A = B = \frac{S - D - C}{2} = \frac{S - 2C}{2}$; conſequently, angle C A V = 90° — D A C, and

A C V = 90° − A C B, are alſo given; and we have in the triangle A C V two angles, and the included ſide A C, to conſtruct the triangle.

THEOREM I.

If the angular points A B C, of the great circle ſpherical triangle A B C, be the poles of three great circles; theſe great circles, by their interſections, will form another triangle F D E, which is called ſupplemental to the former, that is, the ſides F D, D E, E F, are the ſupplements of the meaſures of the oppoſite angles C, B, A, of the triangle A B C, and the meaſures of the angles F, D, E, of the triangle F D E, will be the ſupplements of the ſides A C, B C, B A, in the triangle A B C. — (Pl. 7. fig. 14.)

Let A B, produced, meet D E, E F, in G, M, and A C meet F D, and F E in K L, and B C meet F D, D E, in N, H.

Since A is the pole of F E, and the circle A C paſſes through A, E F will paſs through the pole of A C; and ſince A C paſſes through C, the pole of F D; F D will paſs through the pole of A C, therefore, the pole of A C is in the point F, in which the arches F D, E F, interſect each other. In the ſame manner D is the pole of B C, and E the pole of A B.

And ſince F, E, are the poles of A L, A M, F L, and E M are quadrants, and E L, E M, together, that is, F E and M L together are equal to a ſemicircle. But ſince A is the pole of M L, M L is the meaſure of the angle B A C, conſequently F E is the ſupplement of the meaſure of the angle B A C. In the ſame manner E D, D F, are the ſupplements of the meaſures of the angles A B C, B C A.

Since, likewiſe, C N, B H, are quadrants, C N, B H, together, that is, N H, B C, together, are equal to a ſemicircle; and ſince D is the pole of N H, N H is the meaſure of the angle F D E; therefore, the meaſure of the angle F D E is the ſupplement of the ſide B C. In the ſame manner it is ſhown, that the meaſures of the angles D E F, E F D, are the ſupplements of the ſides A B, A C, in the triangle A B C.

PROBLEM II.

Given the ſides A B, B C, and the ſum of the angles at the baſe A + C, to determine the great circle ſpherical triangle A B C. — (Pl. 7. fig. 8.)

ANALYSIS. Suppoſe the thing done, and about A, B, C, as poles, deſcribe the great circles F E, D E, D F, and produce F D, F E, till they meet in I; then, ſince F E is the ſupplement of the meaſure of angle A, (T. I.) the ſupplement of F E, I E, muſt be the meaſure of angle A. And in like manner we prove that D I is the meaſure of angle C.

Again, angle D E F being the meaſure of the ſupplement of A B, angle D E I muſt be the meaſure of A B; and for the ſame reaſon angle I D E will be found the meaſure of B C. Hence in the triangle A B C, we have given

the angle I E D meaſure of A B,
——— I D E ——— B C,
——— D I + E I ——— A + C.

Compleat the parallelogram D E G H, and join D, G: Becauſe, in the triangle D E G, the angle E is given, the ſum of the angles $EDG+DGE=FI-IDE$, and $GE=IF-DI-IE$; the problem is reduced to the laſt.

PROBLEM III.

Given one ſide AC, the vertical angle A, and the ſum of the remaining ſides AB and BC, to determine the ſpherical triangle ABC, of uniform curvature, &c.—(Pl. 7. fig. 9.)

CON. Having drawn ATc, ABb, including the given angle A, lay off AT, At, each equal $\frac{AC+AB+BC}{2}$; from T, t, erect the perpendiculars TO, Ot to meet in O; with radius $TO=Ot$ and center O, deſcribe the circle TPt, and from A lay off AC, the given ſide. Laſtly, lay off $CP=CT$, with radius CP and center C, which will be a tangent to TP at P, and produced, will cut AB in B.

DEM. By conſtruction, and T. V. B. III. A is the vertical angle, ACB the ſum of the ſides, and AC is the given ſide.

COR. If Ac and angle A had been given,

and $Ab \sim cb$, the difference of the other ſide and baſe, the conſtruction and demonſtration would have been exactly the ſame.

SCHOLIUM. The ſolutions of ſpherical problems may often be facilitated by the method uſed in P. II. Thus, if we ſuppoſe ACB a great circle triangle, (Pl. 7. fig. 8.) AB + BC, AC and angle A given as in the problem, and circles drawn about the poles C, A, B, we have given

DFE = the meaſure of ſupplement AC,
EDF + DEF ——— ſupplement CB + BA,
and FE ——— ſupplement of A,

to conſtruct the triangle DEF, being the ſame data as in P. I.

Or the problem may be conſtructed thus: Suppoſe CB, BA, Pl. 7. fig. 17. produced till they meet in I, and compleat the parallelogram CAPD; join CP, then in the triangle CAP are given CA, AP, $(= BI - \overline{CB + BA})$ and angle A (= ſupplement angle CAB) to determine the reſt.

PROBLEM IV.

Given the vertical angle A, and the

ſum of each ſide ($AC + CQ$ and $AB + BQ$) added to its adjacent ſegment, made by an arch AQO biſecting the vertical angle, to determine the ſpherical triangle. — (Pl. 7. fig. 9.)

ANALYSIS. Since the ſum of each ſide, added to its adjacent ſegment, is given, the perimeter is given, whence making the ſame preparation, as in P. III, the radius $TO = OP$ of the circle TP is alſo given, and ſince $TC = CP$, if we lay off $AV = AC + CQ$, we have $VT = QP$, which laſt will conſequently be given. Again, ſince AT, and the angles TAO, ATO, are all given, TOA is given, and, hence, the following

CON. Having deſcribed the ſmall circle TP, deſcribe a circle CQPB of equal curvature and convexity with AC and AB, to touch it in any point P. Lay off $PQ = AT - AV$, and through O, Q, draw the great circle OQA. From O lay off TO, forming the known angle AOT; and from T draw the circle TA a tangent to TP in T, and cutting CA in A, and CB in C. Draw ABt a tangent to TP in t, (P. IX. B. I.) and cutting CB in B; ſo is CAB the triangle required; the demonſtration of which is plain from the analyſis.

PROBLEM V.

Given the baſe C B, the ſum of the ſides A B + A C, and the vertical angle A, to conſtruct the great circle ſpherical triangle, &c.—(Pl. 7. fig. 16.)

CON. Having laid off the vertical angle A, and drawn the circle T t, as in P. IV, from A lay off L A, V A, each $= \frac{1}{2}.\ \overline{AB + AC - CB}$. Erect the perpendicular arches L Q, V Q, meeting in Q; and with radius L Q = V Q, (C. IV. T. XXII. B. I.) deſcribe the circle L V. Laſtly, draw the circle C I P B, touching the circles L V, T t, in I and P reſpectively, (P. VIII. B. I. of conſtruction) and the thing is done.

DEM. By conſtruction, C A + A B + C B (T. V. B. III.) = the given perimeter, and T A t = the given vertical angle. Alſo T A − L A = T L = B C. (T. VI. B. III.*)

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* This propoſition (and its corollaries) muſt be conſidered as relative to great circle triangles only, for though it involves no abſurdity as there propoſed, it has been diſcovered (ſince it was put to the preſs) that it can only apply in that particular caſe.

PROBLEM VI.

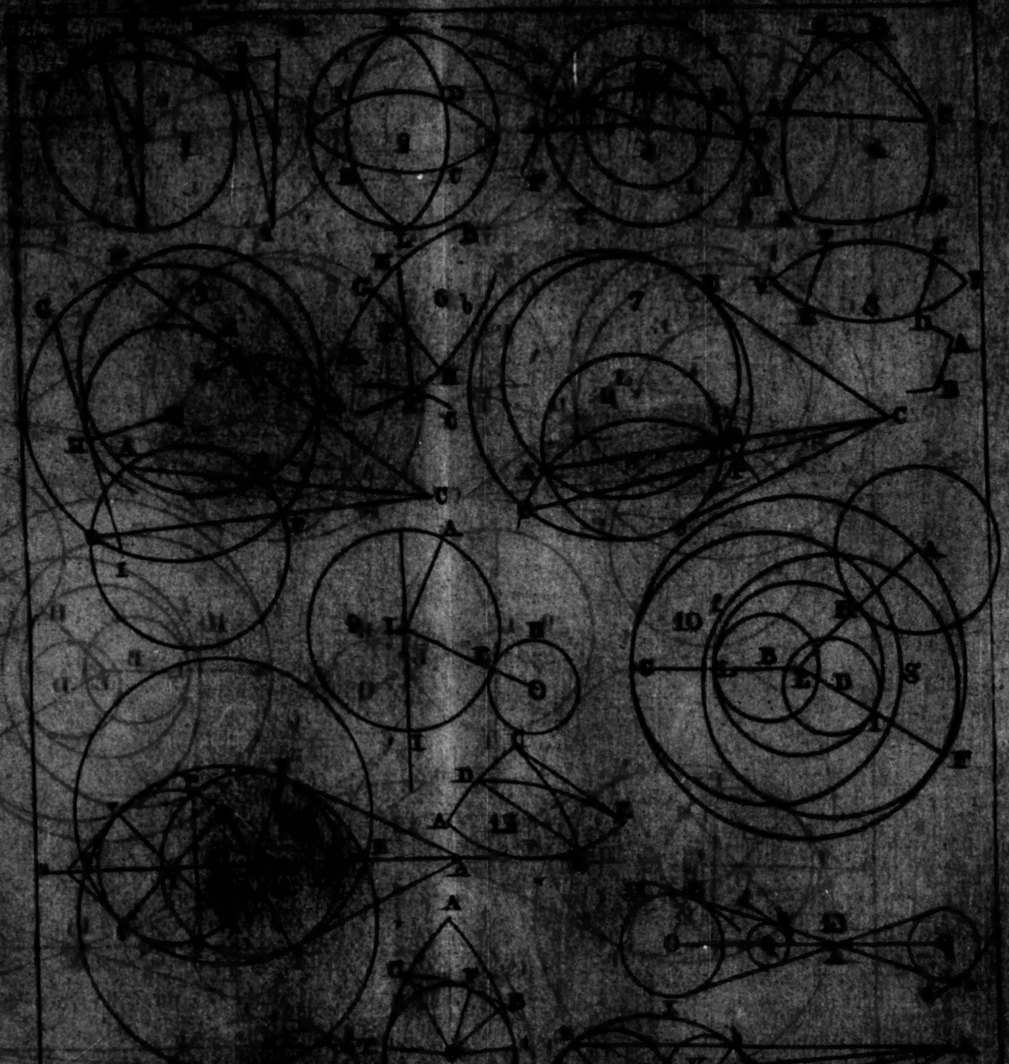
Given the baſe CB, the perpendicular OP, and vertical angle COB, to conſtruct the great circle triangle. — (Pl. 6. fig. 15.)

ANALYSIS. Suppoſe the thing done, and with radius OP, and center O, deſcribe the ſmall circle TPt. Through C and B, draw the great circles ACT, ABt, tangents to TPt in T, t, and meeting in A, and join C, O; B, O; T, O, and tO.

Becauſe PC=TC, and TO=OP, and angle T=P and CO, common to the two triangles TCO, COP, they are equal and ſimilar, and conſequently angle TOC=COP; and ſince in like manner it may be proved that BOt=BOP, therefore the whole angle TOt is equal twice COB, a given quantity.

CON. Hence having deſcribed TPt, and laid off from O the angle TOt, then if tangents TA, tA, be drawn to TPt at T and t, the problem is reduced to the laſt.

Plate 6.



PROBLEM VIII.

Given the vertical angle B V C, and the radii D O, Q E, of two circles inſcribed in the parts B V R, C V R, of a great circle ſpherical triangle B V C, made by a line V R, biſecting that angle to conſtruct the triangle.—(Pl. 7. fig. 13.)

CON. On D O, deſcribe the triangle D V O right angled at D, and having its vertical angle $D V O = \frac{B V C}{4}$, and from V draw V I R a tangent to D I at I parallel to which, (through the center O of D I) draw P O a ſmall circle; join O, I, and upon O I, produced, lay off I L = Q E; and through L draw the ſmall circle L Q parallel to V A R. Again, upon the arch q a = Q E, erect the triangle a q v right angled at a, and having the angle q v a = D V O; and from V apply V A upon V R equal v a. Alſo, at right angles to V A, through A, draw the great circle P A Q, cutting L Q in Q and P O in

P; and with radius AQ and center Q, deſcribe the circle AE. Draw VEC touching AE in E. Laſtly, draw BFGC a tangent to DI and AE in F, G, reſpectively, and meeting VD and VE, produced in B and C reſpectively, and the thing is done.

DEM. By the nature of the circle and conſtruction, angle DVI is half the given angle: Alſo, becauſe PO, VI, QL, are parallels, and the angles at I and A both right angles, we have IO=PA and IL=AQ=QE, the required radii. And, ſince, VA=va and angle AVQ=avq, the point A falls at the required diſtance upon VR from V, the reſt is evident.

PROBLEM VIII.

Given two ſides AC, CB, and the area, to conſtruct the great circle ſpherical triangle.—(Pl. 7. fig. 18.)

When the area is given, the ſum of the angles are alſo given, (T. VIII, B. II.) in which caſe we ſhall have the following

CON. Biſect one ſide CB in Q, and erect the perpendicular QV: Alſo, (by P. I.) draw

C V, meeting Q V in V ſuch, that the ſum of the angles Q C V + C V Q ſhall be equal $\frac{ACB+CAB+ABC}{2}$. Again, (by P. I. B. II.) draw the equal and parallel circles a B V and C I A. Laſtly, with radius C B, the other given ſide and centre C, deſcribe the ſmall circle a B, interſecting a B V in a and B; ſo will C a B or C A B be the required triangle.

DEM. Join V, A, then is the triangle C A B = C V A (C. II. T. V. B. II.) = 2. C Q V, the given area, the reſt is evident.

COR. When C B neither cuts nor touches a V, the problem is impoſſible; and, hence, a and B will coincide when the triangle is a maximum, and C B be perpendicular to both a V and C Q A;

And here occurs an elegant inſtance of the obſervation contained in SCHOLIUM P. III. For having deſcribed about the points A, B, C, as poles, the great circles F E, D E, and D F, (Pl. 7. fig. 8.) then, ſince the ſum of the angles C, A, and B, are given, the ſum of the ſupplement of their meaſures E F + D E + D F, is alſo given; and ſince C A and C B are both given, the meaſures of their reſpective ſupplements, angle D F E, and E D F, are alſo given; Whence the problem may be eaſily and elegantly conſtructed.

PROBLEM IX.

Given the baſe C B, the perpendicular A P, and the area, to conſtruct the ſpherical triangle. — (Pl. 7. fig. 11.)

Here the ſum of the angles A, C, B, being conſequently given, we have this

CON. Having made the triangle C V Q, and drawn the equal parallels A V a, C L B (as in the laſt problem) about the pole I of the great circle C Q B, deſcribe the parallel A E a at the diſtance A I = the complement of the given perpendicular A P to a quadrant; through A, where this interſects A V a in A, draw the great circle I A P, meeting C B (produced if neceſſary) in P; and join A, C, and A, B, and the thing is done.

DEM. It is proved (ſee the laſt problem) that triangle C A B = C V B = the given area; and A P being by conſtruction the complement of A I, is evidently equal to the given perpendicular.

PROBLEM X.

Given the perimeter, the vertical angle V, and the perpendiculars O v, O n, and O I, falling on each ſide B V, V A, and B A, from a point O within the triangle B V A to conſtruct it.—(Pl. 7. fig. 12.)

CON. Having laid off the given vertical angle V, draw b O and O a reſpectively, parallel circles to B V, V A, at the given diſtances of the perpendiculars V O, O n; and from their interſection in O, with radius equal perpendicular O I, deſcribe the circle I E. Laſtly, lay off the circle T P t, of equal perimeters as in P. IV. and draw P I a tangent to both circles at P and I reſpectively (P. VIII. B. I. of conſtruction) and the thing is done. The demonſtration is evident from the conſtruction.

PROBLEM XI.

Given the vertical angle V, and the difference between each ſide and its adjacent ſegment of the baſe made by a perpendicular let fall thereon, from the vertical angle $AV - AP$ and $VB - PB$, to conſtruct the great circle ſpherical triangle AVB. — (Pl. 6. fig. 17.)

PROBLEM XII.

Given the vertical angle V, and the ſum of each ſide added to its adjacent ſegment of the baſe made by a perpendicular VP let fall thereon, from the vertex V; $AV + AP$ and $VB + BP$, to conſtruct the great circle ſpherical triangle AVB. — (Pl. 7. fig. 15.)

☞ The ſolutions of theſe two problems are left to the ingenious.

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The end.

Plate 7.